

Solution for Matrix Equation $AX - YB = C$

Nadia Cahyadi, Yanita, Susila Bahri

Mathematics Postgraduate Study Program,
Faculty of Math and Science, Andalas University,
Kampus UNAND Limau Manis Padang, Indonesia



Abstract - The inverse of a matrix A can only exist if A is nonsingular. This is an important theorem in linear algebra, one learned in an introductory course. In recent years, needs have been felt in numerous areas of applied mathematics for some kind of inverse like matrix of a matrix that is singular or even rectangular. To fulfill this need, mathematicians discovered that even if a matrix was not invertible, there is still either a left or right sided inverse of that matrix. The inverse moore Penrose is an inverse matrix type denoted by $A(1)$. The inverse moore penrose is an extension of the inverse matrix concept. complex matrices will be used to find matrix inverses. Matrix $m \times n$ field F can write as $C_{m \times n}$ with $A(1)$ g-invers of A , the matrix satisfying the equation $AA(1)A=A$. A necessary and sufficient conditions is established for solvability of the matrix equation $AX-YB=C$. Where matrix A, B , and C are giving by equation, we can find the solutions by using Penrose equation existence and construction of $\{1\}$ -inverse to find matrix X and Y satisfying the equation $AX-YB=C$. Substitute the matrix X and the matrix Y to the equation $AX-YB=C$ so that it is proven that the results of $AX-YB$ are matrix C .

Keywords - Penrose Equation, Solution of Linear System.

I. INTRODUCTION

One type of known matrix inverse is generalization inverse. Generalization inverse is an extension of the inverse concept, wherein the inverse matrix is no longer only for the non-singular matrix. There is A matrix generalized inverse, where matrix X is fulfilled if matrix A has an order larger than the non-singular matrix, has ordinary matrix-like properties, and makes it an ordinary inverse if A is a non-singular matrix. In (1955) Penrose first described that for each square or rectangular matrix A with real or complex elements there is a unique X matrix that meets four Penrose equations.

Suppose given a matrix $A \in C_{m \times k}$, $B \in C_{l \times n}$, and $C \in C_{m \times n}$. Next, the solution for the equation will be determined.

$$AX - YB = C$$

which the $\{1\}$ -inverse of each matrix A and B fulfill Penrose equation (1) $AXA = A$. In this paper we will examine the solutions of X and Y of the equation $AX - YB = C$.

II. PENROSE EQUATION

In (1955) Penrose [1] first described that for each square or rectangular matrix A with real or complex elements there is a unique X matrix that meets the following four Penrose equations:

- (1) $AXA = A$
- (2) $XAX = X$
- (3) $(AX)^* = AX$
- (4) $(XA)^* = XA$

Generalized inverse is divided based on the number of Penrose equations that can be met, namely

{1}-inverse, {1,2}-inverse, {1,2,3}-inverse, {1,2,4}-inverse, and {1,2,3,4}-inverse.

III. EXISTENCE AND CONSTRUCTION OF {1}-INVERSE

Theorem 3.1. [1] Suppose $A \in C_{m \times n}$ with $rk(A) = r$, $E \in C_{m \times m}$ and $P \in C_{n \times n}$ are non-singular matrix such that

$$EAP = \begin{bmatrix} I_r & K \\ 0_{(m-r) \times r} & 0_{(m-r) \times (n-r)} \end{bmatrix},$$

where $K \in C_{r \times (n-r)}$, then {1}-inverse of A can be found from following partition matrix

$$A^{(1)} = \begin{bmatrix} I_r & 0_{r \times (m-r)} \\ 0_{(n-r) \times r} & L \end{bmatrix} E.$$

Proof. Suppose that $P \in C_{n \times n}$ and $E \in C_{m \times m}$ are both non-singular matrices, then there is $P^{-1} \in C_{n \times n}$ such that

$$\begin{aligned} AA^{(1)}A &= E^{-1} \begin{bmatrix} I_r & K \\ 0_{(m-r) \times r} & 0_{(m-r) \times (n-r)} \end{bmatrix} P^{-1} P \begin{bmatrix} I_r & 0_{r \times (m-r)} \\ 0_{(n-r) \times r} & L \end{bmatrix} E E^{-1} \begin{bmatrix} I_r & K \\ 0_{(m-r) \times r} & 0_{(m-r) \times (n-r)} \end{bmatrix} P^{-1} \\ &= E^{-1} \begin{bmatrix} I_r & K \\ 0_{(m-r) \times r} & 0_{(m-r) \times (n-r)} \end{bmatrix} P^{-1} \\ &= A \end{aligned}$$

So, it is prove that matrix $A^{(1)}$ is the {1}-inverse of A .

IV. SOLUTION OF LINEAR SYSTEM

In this section we will discuss the application of {1}-inverse which has been obtained from the existence and construction of {1}-inverse on linear system solutions in the case of non-singular matrix.

Theorem 4.1. [1] Suppose that $A \in C_{m \times n}$, $B \in C_{p \times q}$, and $C \in C_{m \times q}$. Then matrix equation.

$$AXB = C, \tag{4.1}$$

Is consistent if and only if, for $A^{(1)}, B^{(1)}$,

$$AA^{(1)}CB^{(1)}B = C, \tag{4.2}$$

or equivalent to

$$AA^{(1)}C = C \text{ and } CB^{(1)}B = C,$$

with general solution

$P^{-1}P = PP^{-1} = I_n$ and $E^{-1} \in C_{m \times m}$ so that $E^{-1}E = EE^{-1} = I_m$. Notice that:

$$\begin{aligned} EAP &= \begin{bmatrix} I_r & K \\ 0_{(m-r) \times r} & 0_{(m-r) \times (n-r)} \end{bmatrix} \\ E^{-1}EAPP^{-1} &= E^{-1} \begin{bmatrix} I_r & K \\ 0_{(m-r) \times r} & 0_{(m-r) \times (n-r)} \end{bmatrix} P^{-1} \\ (E^{-1}E)A(PP^{-1}) &= E^{-1} \begin{bmatrix} I_r & K \\ 0_{(m-r) \times r} & 0_{(m-r) \times (n-r)} \end{bmatrix} P^{-1} \\ A &= E^{-1} \begin{bmatrix} I_r & K \\ 0_{(m-r) \times r} & 0_{(m-r) \times (n-r)} \end{bmatrix} P^{-1} \end{aligned}$$

It will be shown $A^{(1)}$ is the {1}-inverse of A . Note that

$$X = A^{(1)}CB^{(1)} + YA^{(1)}AYBB^{(1)}.$$

for arbitrary $Y \in C_{n \times p}$.

Proof. (\Rightarrow) It will be proved that $AA^{(1)}CB^{(1)}B = C$. Suppose that X fulfills equation (4.1), then then there are $A^{(1)}$ and $B^{(1)}$ which meet Penrose {1}-inverse , so

$$\begin{aligned} AA^{(1)}AXB^{(1)}B &= C. \\ AA^{(1)}CB^{(1)}B &= C \end{aligned}$$

(\Leftarrow) It will be proven that $AXB = C$. Based on equation (4.2) there is $AA^{(1)}CB^{(1)}B$ wich satisfies the equation (4.1). Suppose that $A^{(1)}CB^{(1)} = X$ so

$$\begin{aligned} AA^{(1)}CB^{(1)}B &= C, \\ AXB &= C. \end{aligned}$$

$$\begin{aligned} X &= A^{(1)}CB^{(1)} \\ X_0 &= Y - A^{(1)}AA^{(1)}YB^{(1)}BB^{(1)}, \\ X_0 &= Y - A^{(1)}AYBB^{(1)}. \end{aligned}$$

so that obtained $X_0 = Y - A^{(1)}AYBB^{(1)}$ which satisfies $AXB = 0$. Because $X = A^{(1)}CB^{(1)}$ and $X_0 = Y - A^{(1)}AYBB^{(1)}$, then for each matrix X that meets the equation (4.1) a general solution is obtained, namely

$$X = A^{(1)}CB^{(1)} + YA^{(1)}AYBB^{(1)}.$$

V. SOLUTION OF MATRIX EQUATION $AX - YB = C$

Theorem 5.1. [2] Suppose that $A \in C_{m \times k}$, $B \in C_{l \times n}$, and $C \in C_{m \times n}$. Equation

$$AX - YB = C \tag{5.1}$$

Has solution $X \in C_{k \times n}$, and $Y \in C_{m \times l}$, if and only if

$$(I - AA^{(1)})C(I - B^{(1)}B) = 0, \tag{5.2}$$

The general solution of equation (5.1) has a form

$$X = A^{(1)}C + A^{(1)}ZB + (I - A^{(1)}A)W, \tag{5.3}$$

$$Y = -(I - AA^{(1)})CB^{(1)} + Z - (I - AA^{(1)})ZBB^{(1)}. \tag{5.4}$$

for arbitrary $W \in C_{k \times n}$ and $Z \in C_{m \times l}$.

Proof. (\Rightarrow) It is given $AX - YB = C$, it will be proved necessary conditions $(I - AA^{(1)})C(I - B^{(1)}B) = 0$, of equation (5.1) is fulfilled, by multiplying $(I - AA^{(1)})$ on the left in the two segments equation $AX - YB = C$,

$$\begin{aligned} (I - AA^{(1)})AX - YB &= (I - AA^{(1)})C, \\ AX - YB - AX + AA^{(1)}YB &= C - AA^{(1)}C, \\ AA^{(1)}YB - YB &= C - AA^{(1)}C, \end{aligned}$$

Then by multiplying $(I - BB^{(1)})$ on the right on the two segments, it will become

$$\begin{aligned} (AA^{(1)}YB - YB)(I - BB^{(1)}) & \\ &= (C - AA^{(1)}C)(I - BB^{(1)}), \\ (C - AA^{(1)}C)(I - BB^{(1)}) &= 0, \\ (I - AA^{(1)})C(I - B^{(1)}B) &= 0. \end{aligned}$$

So that it is proven that equation (5.1) satisfies equation (5.2).

(\Leftarrow) To prove the sufficient condition of equation (5.1), given $(I - AA^{(1)})C(I - B^{(1)}B) = 0$, it will be proved $AX - YB = C$. Select $X = A^{(1)}C$ and $Y = -(I - AA^{(1)}CB^{(1)})$, then

$$\begin{aligned} (I - AA^{(1)})C(I - B^{(1)}B) &= 0, \\ C[(I - AA^{(1)})(I - B^{(1)}B)] &= 0, \\ AA^{(1)}C + (I - AA^{(1)})CB^{(1)}B &= C, \\ AX - YB &= C. \end{aligned}$$

Next, it will be proven that the general solution in equations (5.3) and (5.4) fulfills equation (5.5). Use the matrix X and Y given in equations (5.3) and (5.4) to equation (5.1) for each W and Z . To show this matrix is a general solution of equation (5.1), given $X_0 \in C_{k \times n}$ and $Y_0 \in C_{m \times l}$ which one

$$AX_0 - Y_0B = C, \tag{5.5}$$

with $W = X_0$ and $Z = Y_0$ equations (5.3) and (5.4) become

$$\begin{aligned} X &= A^{(1)}C + A^{(1)}ZB + (I - A^{(1)}A)W, \\ &= X_0 - A^{(1)}ZB + (I - A^{(1)}A) \end{aligned}$$

and

$$\begin{aligned} Y &= -(I - AA^{(1)})CB^{(1)} + Z - (I - AA^{(1)})ZBB^{(1)} \\ &= Y_0 - (I - AA^{(1)})(Y_0B + C)B^{(1)}, \end{aligned}$$

Which shows the solution of equation (5.1) that is $X = A^{(1)}C$ and $Y = -(I - AA^{(1)}CB^{(1)})$. So that $X = X_0$ and $Y = Y_0$ fulfill equation (5.5) and equivalent with equation (5.1). So, it is proven that equation (5.3) and equation (5.4) are general solutions of equation (5.1).

So, the solution for the matrix equation $AX - YB = C$ is as follows

$$X = X_0 - A - (AX_0 - Y_0B - C),$$

$$Y = Y_0 - (I - AA^{(1)})(Y_0B + C)B^{(1)},$$

for arbitrary $W \in C_{k \times n}$ and $Z \in C_{m \times l}$.

VI. CONCLUSION

Based on the

1. Generalized inverse is an extension of the inverse concept where the inverse matrix is not only owned by non-singular matrix.
2. The steps in determining the solution for the matrix equation $AX - YB = C$ are as follows
 - 2.1 Determines the $\{1\}$ -inverse of matrices $A^{(1)}$ and $B^{(1)}$ by changing matrix A becomes Hermite's normal form first, so

$$EAP = \begin{bmatrix} I_r & K \\ 0_{(m-r) \times r} & 0_{(m-r) \times (n-r)} \end{bmatrix}$$

and

$$A^{(1)} = \begin{bmatrix} I_r & 0_{r \times (m-r)} \\ 0_{(n-r) \times r} & L \end{bmatrix} E$$

- 2.2 For the equation to have a solution, check whether $\{1\}$ -inverse is the inverse of the matrices A and B based on equation (5.2)

3. If it has fulfilled equation (5.2); it is obtained a general solution of the equation $AX - YB = C$ that is,

$$X = A^{(1)}C + A^{(1)}ZB + (I - A^{(1)}A)W,$$

and

$$Y = -(I - AA^{(1)})CB^{(1)} + Z - (I - AA^{(1)})ZBB^{(1)}.$$

4. Substitute the matrix X and the matrix Y to the equation $AX - YB = C$, so that it is proven that the results of $AX - YB$ are matrix C .

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