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Singular Value Decomposition Guided Image Compression

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Abstract—This paper presents singular value decomposition (SVD), a major technique to matrix decomposition. SVD functions as the fundamental scientific instrument of numerous applications including principal component analysis (PCA), matrix approximation, Eigen Value decomposition, Cholesky decomposition and others. SVD is operated in many applications for example data analysis, Netflix's recommender method, Google's PageRank algorithm, image compression, and dimensional reduction while retaining the most significant information. This paper indicates the mathematics following SVD in a modest way. Furthermore, it applies SVD method in dimensionality reduction and image compression as the essential technique of the data analysis.

Kywords—Data Analysis, Singular Value Decomposition, Dimensionality reduction, Eigen Value Decomposition, Machine Learning

I. INTRODUCTION

Singular Value Decomposition (SVD) is extremely dynamic and broadly matrix decomposition system in computation [1]. It offers the support for almost all data investigation process in data science. SVD is applied to deliver a low rank approximation of several matrices of any structure. SVD approximation is assured to exist and mathematically stable decomposition method. SVD is regarded as the computational mechanism of many information driven algorithm and applications [2]. SVD is exploited in PCA, where high-ranked data is featured into statistically and principal designs of lower rank data. Although SVD is utilized to discover the pseudo inverse of a matrices, and to calculate Pre-determined and less-determined system of linear equations. SVD is often used as the original sources for machine learning model such as classification and clustering [3], [4], signal processing [5], orthogonal decomposition [6], dimensionality reduction algorithm [7], and more.

The residual part of the paper is arranged as follows: section II outlines SVD theory in straightforward way. Section III imparts the connection between SVD and Eigen value decomposition (EVD) via spectral theorem. Section IV clarifies SVD utilization in image compression. Furthermore, section V designates result and discussion of SVD operation as the original computational mechanism in dimensionality reduction treatment. Section VI finalizes the paper.

II. SVD THEORY

In general, a data set is usually represent a structured or unstructured collection of data. Hence, we are considering a large data set $M \in \mathbb{C}^{m\times n}$:

$$M = \begin{pmatrix} m_{1,1} & m_{1,2} & \cdots & m_{1,m} \\ m_{2,1} & m_{2,2} & \cdots & m_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n,1} & m_{n,2} & \cdots & m_{n,m} \end{pmatrix} \tag{1}$$

The columns $m_i \in \mathbb{C}^m$ may be measured from any experiments or simulations of physical prospect. For instance, columns may signify images that have been reformed into column vectors through many entries as pixels into image. SVD is a novel matrix approximation that attend for every complex valued matrix $M \in \mathbb{C}^{m\times n}$. It is denoted by

$$M = P\Sigma Q^T \tag{2}$$

$$= [p_1 \ p_2 \ \dots \ p_k \ \dots \ p_m]\begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_n \end{bmatrix}\begin{bmatrix} {q_1}^T \\ {q_2}^T \\ \vdots \\ {q_k}^T \\ \vdots \\ {q_n}^T \end{bmatrix} \tag{3}$$

Where Q^T is the transpose matrix of $Q, P \in \mathbb{C}^{m\times m}$, $\Sigma \in \mathbb{R}^{m\times n}$, and $Q \in \mathbb{C}^{n\times n}$. The matrices P and Q are square orthogonal matrices of Eigen vectors of A. $Q = [q_1 \ q_2 \dots q_n]$ orthogonally diagonalize $M^T M$. Σ ia a real non-negative diagonal matrix containing singular values are sorted in descending order and the main diagonal elements of Σ are

$\sigma_1 = \sqrt{\lambda_1},\ \sigma_2 = \sqrt{\lambda_2},\ \dots,\sigma_n = \sqrt{\lambda_n}$ referred to as singular values of M where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the nonzero eigen values of $M^T M$ related to the column vectors of Q.

Note that P and Q are also orthonormal matrices and both are defined by:

$$PP^T = P^T P = I_{m\times m}$$

$$QQ^T = Q^T Q = I_{n\times n} \tag{4}$$

The vectors $p_1, p_2, \dots p_n$ are called left singular vectors of M, and the vectors $q_1 \ q_2 \dots q_n$ are called the right singular vectors of M. The rank r of the matrix M is the number of nonzero singular vaues of M.

III. Properties of SVD

Based on row and column numbers, certain properties of M has been exposed in the following cases:

1) when m$\ll n$:in this case the matrix M is a short fat matrix and M may be a full row rank ($k = m$). So, Σ contain at most m non zero singular values on the main diagonal. Whereas Mexactly represent as

$$M = \hat{P}\hat{\Sigma}Q^T \tag{5}$$

where $\hat{P} \in \mathbb{C}^{m\times m}$, $\hat{\Sigma} \in \mathbb{C}^{m\times m}$ and $\tilde{Q} \in \mathbb{C}^{m\times n}$.

Here, $\tilde{P}$ and $\tilde{Q}$ have the first $k \le n$ columns of P and Q respectively and $\tilde{\Sigma}$ contain first $k \times k$ block of Σ. Now in the form of Economy SVD, if k$< m$ it is possible to approximate

$$M \text{ as } M \approx \tilde{P}\tilde{\Sigma}\tilde{Q}^T \tag{6}$$

where $\tilde{P} \in \mathbb{C}^{m\times k}$, $\tilde{\Sigma} \in \mathbb{C}^{k\times k}$ and $\tilde{Q} \in \mathbb{C}^{k\times n}$.

2) when $m \gg n$: in this case the matrix M is a tall-skinny matrix and M may be a full column ($k = n$) and sigma matrix contain at most n nonzero singular values. Whereas M may represent as

$$M = \hat{P}\hat{\Sigma}Q^T \tag{7}$$

where $\hat{P} \in \mathbb{C}^{m\times n}$, $\hat{\Sigma} \in \mathbb{C}^{n\times n}$. Now in the form of Economy SVD, if k$< n$ it is possible to approximate

$$M \text{ as } M \approx \tilde{P}\tilde{\Sigma}\tilde{Q}^T \tag{8}$$

where $\tilde{P} \in \mathbb{C}^{m\times k}$, $\tilde{\Sigma} \in \mathbb{C}^{k\times k}$ and $\tilde{Q} \in \mathbb{C}^{k\times n}$.

3) when $m = n$: in this case matrix M is a square matrix and Eigenvalue Decomposition (EVD) is used for this square matrix as follows:

$$M = ADA^{-1} \quad (9)$$

where $A \in \mathbb{C}^{m\times m}$ is a Eigen vector matrix of M and $D \in \mathbb{C}^{m\times m}$ is a diagonal matrix which contain eigenvalues. Generally, columns of A are linearly independent vectors. These vectors are not orthonormal. In symmetric matrix, $M = M^T$, approximation of M is given by $M = BDB^T$ (10)

which is known as spectral theorem of M.

Thus, B is a square orthogonal and orthonormal matrix of eigenvectors of M.

Note that B can express as follows:

$B^T B = BB^T = I$ and $B^{-1} = B^T$.

IV. Image Compression using SVD

As we have mentioned above, when SVD is applied along a given image matrix such as M where each entry in the matrix corresponds to a pixel value, then it is decomposed into three different matrices P, Σ and Q as it mentioned earlier. By selecting only the largest singular values and corresponding vectors from (P) and(Q), a compressed approximation of (M) can be formed. This reduce the amount of data needed to store the image. Nevertheless, image does not compress it only utilizing SVD. SVD enables the representation of an image using fewer values by capturing the essence of an image in a smaller set of singular values. This is crucial for compressing images without significant loss of quality. The rank of the matrix after SVD reflects the number of singular values that are needed to represent the image. From the definition of SVD in equation (2) and from the properties of SVD, components of SVD for the matrix M can be expressed as sum of k rank-1 matrix as

$$M = \sum_{i=1}^{r} \sigma_i \, p_i {q_i}^T \quad (11)$$

where $\sigma_i$is the ith singular values of matrix M, p_i, q_i are the corresponding singular vector matrices of M and $r = \min(m, n)$.

Equation (11) can be rewritten as,

$$M = \sigma_1 p_1 {q_1}^T + \sigma_2 p_2 {q_2}^T + \cdots + \sigma_n p_n {q_n}^T \quad (12)$$

where r is the rank of M.

Above formula indicates that first term of the summation would have largest contribution and the last term would have the smallest contribution. To achieve largest amount of compression, equation (12) becomes

$$M_k = \sigma_1 p_1 {q_1}^T + \sigma_2 p_2 {q_2}^T + \cdots + \sigma_k p_k {q_k}^T \quad (13)$$

where $k < r$.

Using equation (13), reconstructed image will reduce the storage space requirement to $k(m + n + 1)$ bytes as compared to storage space requirement of mn bytes of the main uncompressed image.

To achieve required storage space, we can simplify (mn) matrix with the values ranging from 0 to 255.

In short, value of k be picked in such a way that better amount of compression is attained.

By comparing the rank and information stored we can assess the compression ratio and the effectiveness of the compression. A lower rank may means that the image can be compressed more, reducing storage requirements without significant affect visual quality. To evaluate the outcomes of the various compression technique and as well as to measure the storage degree to which a figure is compressed, many assessment measures are follows:

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1. Compression Ratio: This is calculated by the comparing the size of the original data to the size of the compressed data and is defined by

$$\frac{mn}{k(m+n+1)}.$$

Hence, Space required (%) is $\frac{(m+n+1)k}{mn} \times 100$.

2. Information stored (%): This is indicated by the ratio of the sum of the square of the singular values of the diagonal matrix to the sum of the square of total singular values of the diagonal matrix. Mathematically it can be written as,

$= (\frac{\sum_{i=1}^{k} \sigma_i{}^2}{\sum_{i=1}^{n} \sigma_i{}^2}) \times 100.$

3. Frobenius norm: The Frobenius norm of a matrix is defined as square root of the sum of the square of its singular values. In SVD based image compression, the Frobenius norm of the difference between the original matrix M and its approximation M_k is defined by, $\|M - M_k\|_F = \sqrt{\sum_{i,j}^{m,n} | m_{ij} - m_{ij}{}^k |^2}$. Frobenius norm indicates the root mean square error between original image and its approximate.

V. RESULT AND DISCUSSION

A colorful beach image is considered for compression whose original size is 259×194 shown in figure-1(a). We can approximate the image by using only the first k singular values and Python program is used to complete this work. This approximation retains the essential features of the image while reducing storage space. For image compression, we choose a value of k (number of singular values) to balance reconstruction quality and compression ratio. In the context of SVD, the Frobenius norm quantifies the difference between the original image matrix and its reconstructed version using a reduced rank. In figure 1(b), rank 5 is considered in which case space required is 4.52% and information stored is 58.17 in percentage but Frobenius norm is 15.41 with lower image fidelity. In figure 1(c), with rank 25 space required is 22.59% and information stored is 76.28 in percentage but Frobenius norm is 8.12. In this case, image quality for rank 25 is better than for rank 5. In this case space required and information storage retained is larger than for rank 5 but Frobenius norm is smaller means less error. For rank 50 in figure 1(d), space required percentage is 45.18, information stored percentage is 86.81 but Frobenius norm is 4.86. Hence, root mean square error is decreased while space required (%) and information stored (%) is increased. Similarly, In figure 1(e) and 1(f) with rank 75 and 100, achieved space required is 67.77% and 90.36% and Information stored is 93.24% & 97.28% respectively. But both cases Frobenius norm is 2.83 and 1.35 sequentially. It is clearly seen that image quality is getting better while increasing the number of singular values. Each case space required and information stored is increased but Frobenius error is decreased. So, larger singular values preserve better image quality but require more storage. Cumulative energy kept in top k singular values by comparing rank with cumulative sum of the singular values of sigma matrix in figure 2(a). In figure 2(b), Comparison between log sigma versus rank (singular values) allows us to emphasize the important features while suppressing noise and less relevant details. Using the logarithm of singular values against the rank helps strike a balance between compression efficiency and image quality. Figure 2(c) indicates Frobenius norm is decreasing while number of singular

value increasing. By comparing rank versus percentage storage required in figure 2(d), can be judged that higher ranks preserve more information but require larger storage space.

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Figure-1(a)

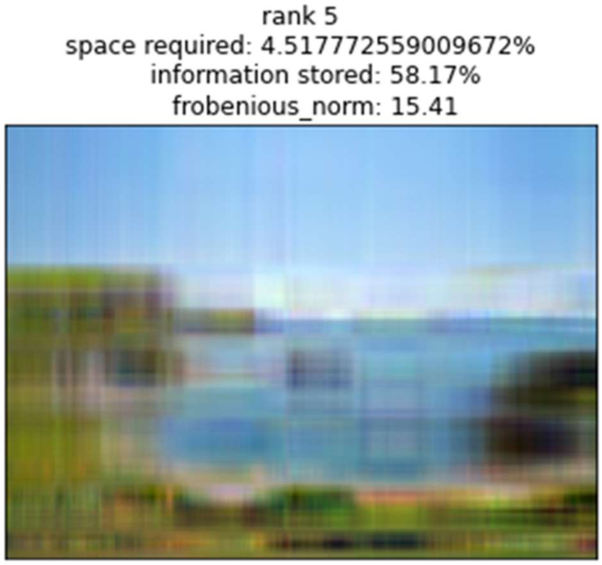

Figure-1(b)

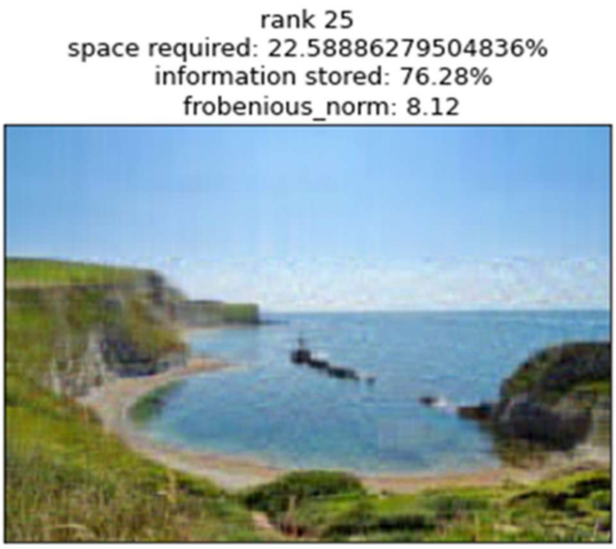

Figure-1(c)

Figure-1(d)

Figure-1(e)

Figure-1(f)

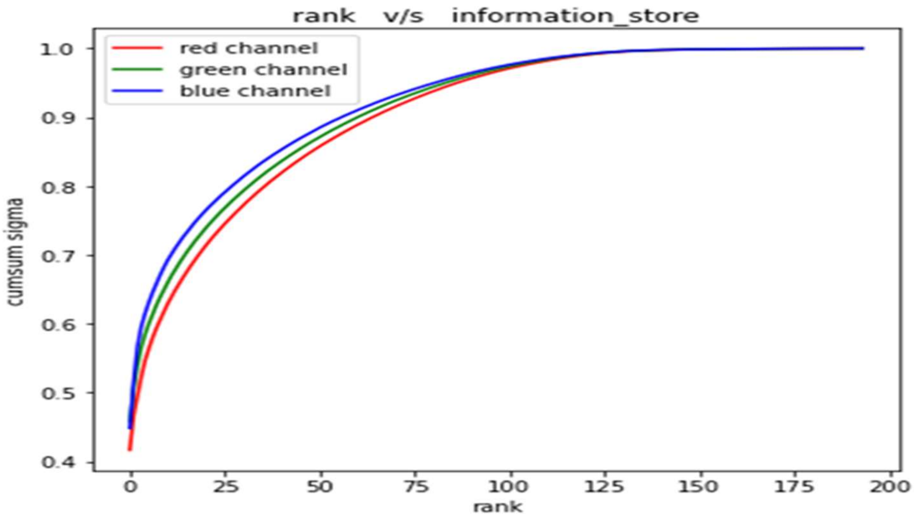

Figure-2(a)

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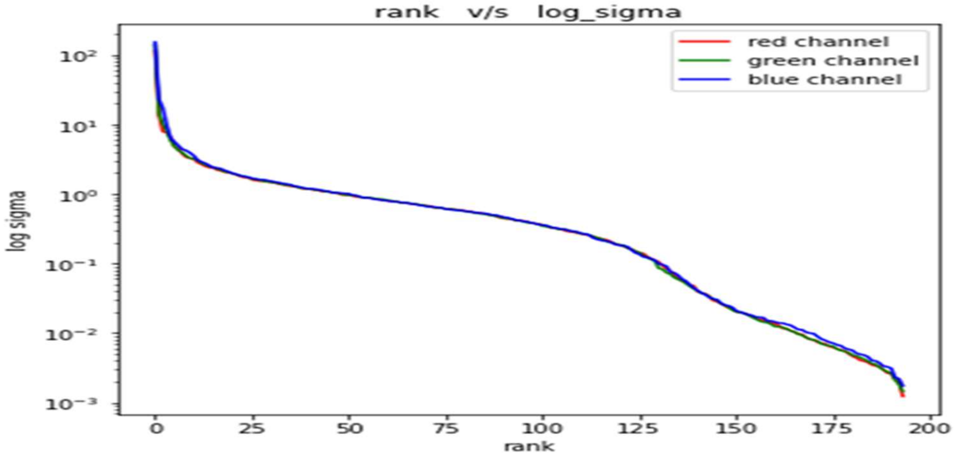

Figure-2(b)

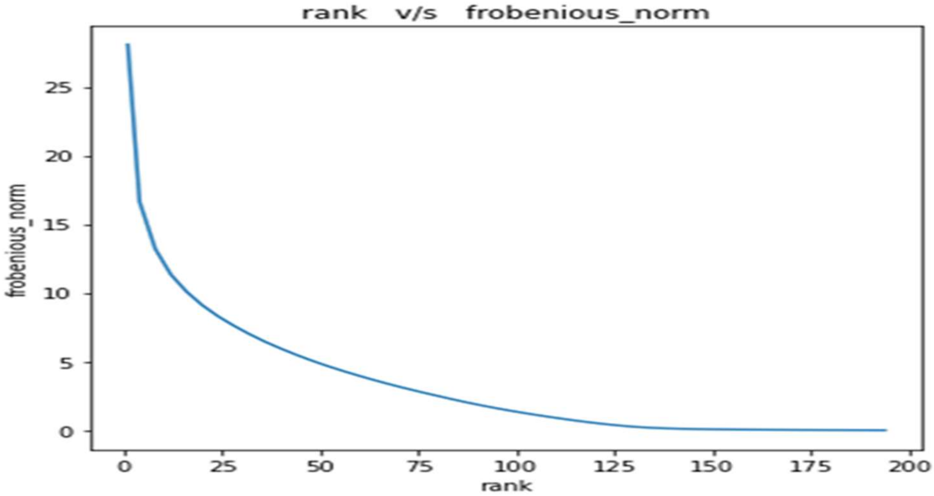

Figure-2(c)

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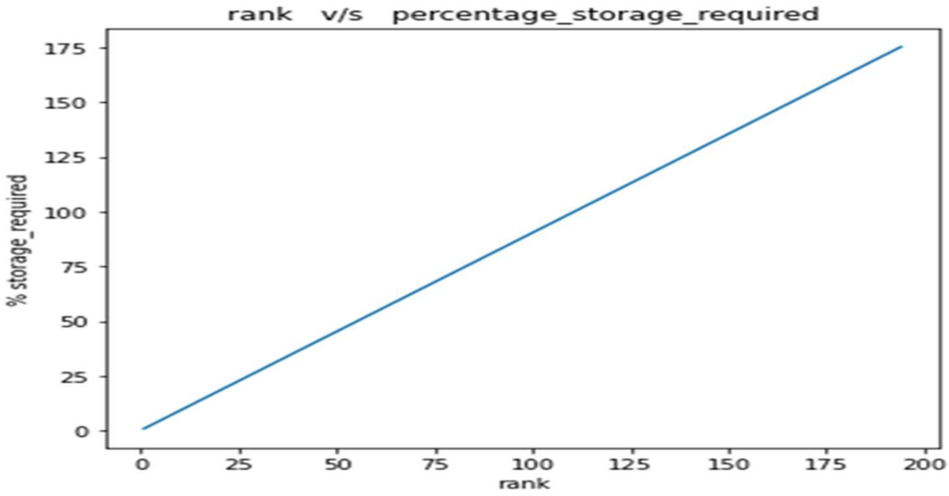

Figure-2(d)

VI. CONCLUSION

This paper demonstrate a modest outline to Singular Value Decomposition (SVD) method. SVD is a foundation and significant tool for various techniques and algorithms especially in machine learning. SVD has a great application in practical life including the field of data science, artificial intelligence, linear system, dynamical system, and so on. Singular Value Decomposition shows how it can work in image compression by retaining the largest and significant singular values in an image for dimensionality reduction. By using fewer singular values, we represented the necessary features of the image while compressing it. Space percentage, Information storage percentage, and Frobenius norm are carried out and also shown their comparison based on overall image compression in context of SVD. The compressed image requires less storage space compared to the original image. In the reconstructed image, image quality getting better follows that required space (%) and Information storage (%) taking larger while retaining k-largest singular values.

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The work is also original except where indicated by and attached with special reference in the context.

REFERENCES

[1] Anowar, F., Sadaoui, S., & Selim, B. (2021). “Conceptual and empirical comparison of dimensionality reduction algorithms (PCA, KPCA, LDA, MDS, SVD, LLE, ISOMAP, LE, ICA, t-SNE),” Computer Science Review, 40, 100378.

[2] Tanwar, S., Ramani, T., & Tyagi, S. (2017). “Dimensionality reduction using pca and svd in big data: A comparative case study,” International conference on future internet technologies and trends, Springer, 116–125.

[3] Masoud, M., Jaradat, Y., Rababa, E., and Manasrah, A. (2021). “Turnover prediction using machine learning: Empirical study,” International Journal of Advance Software Computer Application, 13(1).

[4] Wang, Y., & Zhu, L. (2017). “Research and implementation of svd in machine learning,” IEEE/ACIS 16th International Conference on Computer and Information Science (ICIS), 471–475.

[5] Nie, Z., & Zhao, X. (2016). “Similarity of signal processing effect between pca and svd and its mechanism analysis,” Journal of vibration and shock, 35(2), 12–17.

[6] Schmidt, O. T., & Colonius, T. (2020). "Guide to spectral proper orthogonal decomposition," Aiaa journal, vol. 58(3), 1023–1033.

[7] Bai, Z., Kaiser, E., Proctor, J. L., Kutz, J. N., & Brunton, S. L. (2020). "Dynamic mode decomposition for compressive system identification," AIAA Journal, 58(2), 561–574.

[8] Anton, H., & Rorres, C. (2019). Elementary Linear Algebra, Application version, Wiley, USA, 11th Edition.

[9] Brunton, S. L., & Kutz, J. N. (2019). Data Driven Science & Engineering Machine Learning, Dynamical Systems, and Control, Cambridge University Press, UK.

[10] Cunningham J. P., & Ghahramani, Z. (2015). "Linear dimensionality reduction: Survey, insights, and generalizations," The Journal of Machine Learning Research, 16(1), 2859–2900.