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On The Relationship Between The Matrix Operators of $vech^*$ and $vecp^*$

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Abstract— **This article discusses two new matrix operators constructed differently from $vech$ and $vecp$ by taking a square matrix's main diagonal and supra-diagonal entries. We call these two operators the $vech^*$ and $vecp^*$. We explicitly construct a matrix that transforms $vech^*(A)$ to $vecp^*(A)$, where A is an $n \times n$ matrix for $n = 2, 3, .., 6$. We also derive various properties from the matrix.**

Keywords— ***permutation matrix; vecp*; vech*; vec operator***

I. Introduction

The vec operator is a unique operation on a matrix that transforms the matrix into a column vector. There is another operator defined by [1], which is called the $vech$ operator. The $vech$ operator is the operator by eliminating the supra-diagonal entries. The study of the application vec and $vech$ was carried out by [1], with a focused study on the symmetric matrix in developing multivariate statistical results. Furthermore, other applications of vec and $vech$ were carried out by [2], referred to [1], to obtain a generalization of some of the matrix results given by [3]. In particular, the variance-covariance matrix of the Wishart distribution is obtained in a very compact nonsingular form.

Another relationship between the vec operator related to the Kronecker product and the vec-permutation matrix can be seen in [4, 5]. In addition, another vec operator is defined which is called $vecb$ by [6, 7] related to the block matrix and the Kronecker product. The vec operator is also associated with several particular matrices, such as the permutation matrix, the commutation matrix and the duplication matrix. These matrices are a matrix that transforms vec operator to $vech$, $vecp$ or $vecd$ operator (see [8, 9, 10]).

The method in this study is a literature study. The first step of this research is to define the $vech^*$ and $vecp^*$ for arbitrary $n \times n$ matrix (see Section III). Next, define the unique matrix associated with $vech^*$ and $vecp^*$, and find its properties.

II. Basic theory

This section presents the definitions, properties, and theorems used in this article.

Definition 2.1 [8] *Let $A = [a_{ij}]$ be an $m \times n$ matrix, and A_j is the jth column of A. The $vec(A)$ is the $mn \times 1$ vector gives by*

$$vec(A) = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{pmatrix}.$$

Definition 2.2 [1, 11] *Let* $A = [a_{ij}]$ *be an* $n \times n$ *matrix. The* $vech(A)$ *is the* $\frac{1}{2}n(n+1) \times 1$ *vector that is obtained from* $vec(A)$ *by eliminating all supra-diagonal elements of A.*

For example, for $n = 2$,

$vec(A) = (a_{11}, a_{21}, a_{12}, a_{22})^T$ and $vech(A) = (a_{11}, a_{21}, a_{22})^T$.

Definition 2.3 [9] *Let* $A = [a_{ij}]$ *be an* $n \times n$ *matrix. The* $vecp(A)$ *is defined to be a column vector consisting of the lower triangular elements of A, it is given as*

$$vecp(A) = \left(a_{11}, a_{22}, \dots, a_{nn}, a_{21}, a_{31}, \dots, a_{n1}, a_{32}, a_{42}, \dots, a_{n2}, \dots, a_{n,n-1}\right)^T.$$

For example, for $n = 3$,

$vecp(A) = (a_{11}, a_{22}, a_{33}, a_{21}, a_{31}, a_{32})^T$.

Let S_n denote the set of all permutations of the n element set $[n] \coloneqq \{1,2,\dots,n\}$. A permutation is a one-to-one function from $[n]$ onto $[n]$. The permutation of finite sets is usually given by listing each domain element and its corresponding functional value. For example, we define a permutation σ of the set $[n] \coloneqq \{1,2,3,4,5,6\}$ by specifying $\sigma(1) = 5$, $\sigma(2) = 3$, $\sigma(3) = 1$, $\sigma(4) = 6$, $\sigma(5) = 2$, $\sigma(6) = 4$. A more convenient way to express this correspondence is to write σ in array form as

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 1 & 6 & 2 & 4 \end{pmatrix} \tag{2.1}$$

There is another notation commonly used to specify permutation. It is called cycle notation. For example, permutation in (2.1) can be written as $\sigma = (1\ 5\ 2\ 3)(4\ 6)$. For detail, see [12].

If σ is a permutation, we have the identity matrix as follows:

Definition 2.4 [13] *Let* σ *be a permutation in* S_n. *Define the permutation matrix* $P(\sigma) = \left(\delta_{i,\sigma(j)}\right)$, $\delta_{i,\sigma(j)} = entry_{i,j}\left(P(\sigma)\right)$ *where*

$$\delta_{i,\sigma(j)} = \begin{cases} 1 & \text{if } i = \sigma(j) \\ 0 & \text{if } i \neq \sigma(j) \end{cases}$$

Example 2.1 Let $n \coloneqq \{1,2,3\}$ and $\sigma = (1\ 2\ 3)$.

$P(123) = [\delta_{i,\sigma(j)}]$ and $\delta_{i,\sigma(j)} = \begin{cases} 1 \text{ if } i = \sigma(j) \\ 0 \text{ if } i \neq \sigma(j) \end{cases}$

(1 to 2; 2 to 3; 3 to 1; $\sigma(1) = 2, \sigma(2) = 3, \sigma(3) = 1$)

$ent_{11}\left(P(\sigma)\right) = \delta_{1,\sigma(1)} = 0\ (\sigma(1) = 2)$; $ent_{12}\left(P(\sigma)\right) = \delta_{1,\sigma(2)} = 0\ (\sigma(2) = 3)$; $ent_{13}\left(P(\sigma)\right) = \delta_{1,\sigma(3)} = 1\ (\sigma(3) = 1)$;

$ent_{21}\left(P(\sigma)\right) = \delta_{2,\sigma(1)} = 1\ (\sigma(1) = 2)$; $ent_{22}\left(P(\sigma)\right) = \delta_{2,\sigma(2)} = 0\ (\sigma(2) = 3)$; $ent_{23}\left(P(\sigma)\right) = \delta_{2,\sigma(3)} = 0\ (\sigma(3) = 1)$;

$ent_{31}\left(P(\sigma)\right) = \delta_{3,\sigma(1)} = 0\ (\sigma(1) = 2)$; $ent_{32}\left(P(\sigma)\right) = \delta_{3,\sigma(2)} = 1\ (\sigma(2) = 3)$; $ent_{33}\left(P(\sigma)\right) = \delta_{3,\sigma(3)} = 0\ (\sigma(3) = 1)$.

So we have $P(123) = \begin{pmatrix} \delta_{12} & \delta_{13} & \delta_{11} \\ \delta_{22} & \delta_{23} & \delta_{21} \\ \delta_{32} & \delta_{33} & \delta_{31} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

We present definitions of inversion on permutation and elementary product.

Definition 2.5 [14] *Inversion is the occurrence of a larger integer preceding a smaller integer, while the number of inversions is the total number of integers preceded by a smaller integer in each inversion in the order of the permutations.*

Definition 2.6 [14] *If the number of inversions of a permutation is an even number then it is said to be an even permutation, and if it is an odd number, then it is said to be an odd permutation.*

Definition 2.7 [14] *Let A be an $n \times n$ matrix. The elementary product of A is the product of n elements from A without taking elements from the same row or column, while the signed elementary product of A is the elementary product which is marked $(+1)$ if the permutation is even and (-1) if the permutation is odd.*

We also present the definition of the orthogonal matrix and the relationship between the permutation matrix and the orthogonal matrix.

Definition 2.8 [8] *An $m \times m$ matrix P whose columns form an orthonormal set of vectors is called an orthogonal matrix. It immediately follows that $P^T P = PP^T = I_m$.*

Theorem 2.1 [8] *Let P be $m \times m$ orthogonal matrix. Then $|P| = \pm 1$, so that P is nonsingular. Consequently, $P^{-1} = P^T$.*

Theorem 2.2 [11] *Every permutation matrix is an orthogonal matrix.*

III. Results and Discussion

The aim of this paper to introduce new operator like $vech$ and $vecp$ and we called the operator with $vech^*$ and $vecp^*$, and then to presented the relationship between $vech^*$ and $vecp^*$.

Definition 3.1 *Let $A = [a_{ij}]$ be an $n \times n$ matrix. The $vech^*(A)$ is the $\frac{1}{2}n(n+1) \times 1$ vector that is obtained from $vec(A)$ by eliminating all below main diagonal elements of A, i.e.:*

$$vech^*(A) = (a_{11}, a_{12}, a_{22}, a_{13}, a_{23}, a_{33}, \ldots, a_{1n}, a_{2n}, \ldots, a_{nn})^T.$$

Definition 3.2 *Let $A = [a_{ij}]$ be an $n \times n$ matrix. The $vecp^*(A)$ is the $\frac{1}{2}n(n+1) \times 1$ vector that stacks the main diagonal elements and then the supra-diagonal elements in order of the first column to the last column of A, i.e.:*

$$vecp^*(A) = \left(a_{11}, a_{22}, a_{33}, \ldots, a_{nn}, a_{12}, a_{13}, a_{23}, \ldots, a_{1n}, \ldots, a_{n-1,n}\right)^T.$$

Example 3.1 Suppose given a matrix A of size 4×4 as follows:

$$A = \begin{pmatrix} 2 & 0 & 1 & 8 \\ 1 & 8 & 3 & 2 \\ 5 & 8 & 3 & 6 \\ 1 & 9 & 4 & 7 \end{pmatrix}$$

Then

$vech^*(A) = (2, 0, 8, 1, 3, 3, 8, 2, 6, 7)^T$ and $vecp^*(A) = (2, 8, 3, 7, 0, 1, 3, 8, 2, 6)^T$.

Let A *be an* $n \times n$ *matrix*. In [9], it is stated that there is an $n \times n$ matrix B_n^p that transforms $vech(A)$ to $vecp(A)$, i.e: $B_n^p vech(A) = vecp(A)$. In this paper, will construct a matrix similar to B_n^p, symbolized by $B_n^{p^*}$, which transforms $vech^*(A)$ to $vecp^*(A)$, i.e., $B_n^{p^*} vech^*(A) = vecp^*(A)$. The $B_n^{p^*}$ matrix to be constructed for $n = 2, 3, 4, 5, 6$.

a. For $n = 2$. Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, and we have $B_2^{p^*} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, i.e.:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} vech^*(A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} (a_{11}, a_{12}, a_{22})^T = (a_{11}, a_{22}, a_{12})^T = vecp^*(A)$$

b. For $n = 3$. Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, and we have $B_3^{p^*} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$, i.e.:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} vech^*(A) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{12} \\ a_{22} \\ a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{22} \\ a_{33} \\ a_{12} \\ a_{13} \\ a_{23} \end{pmatrix} = vecp^*(A)$$

c. For $n = 4$. Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$, and we have $B_4^{p^*} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$, i.e.:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} vech^*(A) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{12} \\ a_{22} \\ a_{13} \\ a_{23} \\ a_{33} \\ a_{14} \\ a_{24} \\ a_{34} \\ a_{44} \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{22} \\ a_{33} \\ a_{44} \\ a_{12} \\ a_{13} \\ a_{23} \\ a_{14} \\ a_{24} \\ a_{34} \end{pmatrix} = vecp^*(A)$$

d. For $n = 5$. Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{pmatrix}$, and we have $B_5^{p^*} = \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \end{pmatrix}$,

i.e.:

$$\begin{pmatrix} 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \end{pmatrix} vech^*(A) = \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{12} \\ a_{22} \\ a_{13} \\ a_{23} \\ a_{33} \\ a_{14} \\ a_{24} \\ a_{34} \\ a_{44} \\ a_{15} \\ a_{25} \\ a_{35} \\ a_{45} \\ a_{55} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} \\ a_{22} \\ a_{33} \\ a_{44} \\ a_{55} \\ a_{12} \\ a_{13} \\ a_{23} \\ a_{14} \\ a_{24} \\ a_{34} \\ a_{15} \\ a_{25} \\ a_{35} \\ a_{45} \end{pmatrix} = vecp^*(A)$$

e. For $n = 6$. Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{pmatrix}$, and we have

$$B_6^{p^*} = \begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0
\end{pmatrix}, \text{ i.e.:}$$

$$\begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0
\end{pmatrix} vech^*(A) =$$

$$\begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0
\end{pmatrix}
\begin{pmatrix} a_{11}\\ a_{12}\\ a_{22}\\ a_{13}\\ a_{23}\\ a_{33}\\ a_{14}\\ a_{24}\\ a_{34}\\ a_{44}\\ a_{15}\\ a_{25}\\ a_{35}\\ a_{45}\\ a_{55}\\ a_{16}\\ a_{26}\\ a_{36}\\ a_{46}\\ a_{56}\\ a_{66} \end{pmatrix} =
\begin{pmatrix} a_{11}\\ a_{22}\\ a_{33}\\ a_{44}\\ a_{55}\\ a_{66}\\ a_{12}\\ a_{13}\\ a_{23}\\ a_{14}\\ a_{24}\\ a_{34}\\ a_{15}\\ a_{25}\\ a_{35}\\ a_{45}\\ a_{16}\\ a_{26}\\ a_{36}\\ a_{46}\\ a_{56} \end{pmatrix} = vecp^*(A)$$

Consider that for $n = 2$, size of $B_2^{p^*}$ is 3×3, for $n = 3$, $B_3^{p^*}$is 6×6, for $n = 4$, $B_4^{p^*}$ is 10×10, for $n = 5$, $B_5^{p^*}$ is 15×15, and for $n = 6$, $B_6^{p^*}$ is 21×21. Based on this, we have the size of $B_n^{p^*}$, $n = 2, 3, 4, 5, 6$ is $\frac{n(n+1)}{2} \times \frac{n(n+1)}{2}$ (This can be proven by mathematical induction).

Next, the form of $B_n^{p^*}$, $n = 2, 3, 4, 5, 6$ will be written in a formula. We need several symbols, i.e.:

- $\boldsymbol{e}_{1,n}$ is a row matrix containing one element 1 in the first column
- $O_{m \times n}$ is a zero matrix consisting of m-row and n-column
- $F_n = \left[O_{(n-1)\times 1}, I_{n-1}\right]$

Thus form $B_n^{p^*}$, $n = 2, 3, 4, 5, 6$ is written as follows:

- $B_2^{p^*} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{e}_{1,2}^T & O_{1\times 1} \\ O_{1\times 2} & 1 \\ O_{1\times 1} & \boldsymbol{e}_{1,2}^T \end{pmatrix}$

- $B_3^{p^*} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{e}_{1,3}^T & O_{1\times 1} & O_{1\times 2} \\ O_{1\times 2} & \boldsymbol{e}_{1,2}^T & O_{1\times 2} \\ O_{1\times 3} & O_{1\times 2} & \boldsymbol{e}_{1,1}^T \\ O_{1\times 1} & \boldsymbol{e}_{1,3}^T & O_{1\times 2} \\ O_{2\times 2} & F_3 & O_{2\times 1} \end{pmatrix}$

- $B_4^{p^*} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{e}_{1,4}^T & O_{1\times 1} & O_{1\times 2} & O_{1\times 3} \\ O_{1\times 2} & \boldsymbol{e}_{1,3}^T & O_{1\times 2} & O_{1\times 3} \\ O_{1\times 3} & O_{1\times 2} & \boldsymbol{e}_{1,2}^T & O_{1\times 3} \\ O_{1\times 4} & O_{1\times 3} & O_{1\times 2} & \boldsymbol{e}_{1,1}^T \\ O_{1\times 1} & \boldsymbol{e}_{1,4}^T & O_{1\times 2} & O_{1\times 3} \\ O_{2\times 2} & F_3 & O_{2\times 1} & O_{2\times 4} \\ O_{3\times 3} & O_{3\times 2} & F_4 & O_{3\times 1} \end{pmatrix}$

- $B_5^{p^*} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{e}_{1,5}^T & O_{1\times 1} & O_{1\times 2} & O_{1\times 3} & O_{1\times 4} \\ O_{1\times 2} & \boldsymbol{e}_{1,4}^T & O_{1\times 2} & O_{1\times 3} & O_{1\times 4} \\ O_{1\times 3} & O_{1\times 2} & \boldsymbol{e}_{1,3}^T & O_{1\times 3} & O_{1\times 4} \\ O_{1\times 4} & O_{1\times 3} & O_{1\times 2} & \boldsymbol{e}_{1,2}^T & O_{1\times 4} \\ O_{1\times 5} & O_{1\times 4} & O_{1\times 3} & O_{1\times 2} & \boldsymbol{e}_{1,1}^T \\ O_{1\times 1} & \boldsymbol{e}_{1,5}^T & O_{1\times 2} & O_{1\times 3} & O_{1\times 4} \\ O_{2\times 2} & F_3 & O_{2\times 1} & O_{2\times 5} & O_{2\times 4} \\ O_{3\times 3} & O_{3\times 2} & F_4 & O_{3\times 1} & O_{3\times 5} \\ O_{4\times 4} & O_{4\times 3} & O_{4\times 2} & F_5 & O_{4\times 1} \end{pmatrix}$

- $$B_6^{p^*} = \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\ 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{e}_{1,6}^T & O_{1\times1} & O_{1\times2} & O_{1\times3} & O_{1\times4} & O_{1\times5}\\ O_{1\times2} & \boldsymbol{e}_{1,5}^T & O_{1\times2} & O_{1\times3} & O_{1\times4} & O_{1\times5}\\ O_{1\times3} & O_{1\times2} & \boldsymbol{e}_{1,4}^T & O_{1\times3} & O_{1\times4} & O_{1\times5}\\ O_{1\times4} & O_{1\times3} & O_{1\times2} & \boldsymbol{e}_{1,3}^T & O_{1\times4} & O_{1\times5}\\ O_{1\times5} & O_{1\times4} & O_{1\times3} & O_{1\times2} & \boldsymbol{e}_{1,2}^T & O_{1\times5}\\ O_{1\times6} & O_{1\times5} & O_{1\times4} & O_{1\times3} & O_{1\times2} & \boldsymbol{e}_{1,1}^T\\ O_{1\times1} & \boldsymbol{e}_{1,6}^T & O_{1\times2} & O_{1\times3} & O_{1\times4} & O_{1\times5}\\ O_{2\times2} & F_3 & O_{2\times1} & O_{2\times6} & O_{2\times5} & O_{2\times4}\\ O_{3\times3} & O_{3\times2} & F_4 & O_{3\times1} & O_{3\times6} & O_{3\times5}\\ O_{4\times4} & O_{4\times3} & O_{4\times2} & F_5 & O_{4\times1} & O_{4\times6}\\ O_{5\times5} & O_{5\times4} & O_{5\times3} & O_{5\times2} & F_6 & O_{5\times1} \end{pmatrix}$$

Next, the properties associated with $B_n^{p^*}, n = 2, 3, 4, 5, 6$ are as follows:

Theorem 3.1 *The* $B_n^{p^*}, n = 2, 3, 4, 5, 6$ *is a permutation matrix.*

Proof. Based on Definitions 2.4 and 3.3, the $B_n^{p^*}, n = 2, 3, 4, 5, 6$ is a permutation matrix.

■

Corollary 3.1 *The* $B_n^{p^*}, n = 2, 3, 4, 5,6$ *is an orthogonal matrix.*

Proof. Based on Theorem 3.1 obtained that $B_n^{p^*}, n = 2, 3, 4, 5, 6$ is a permutation matrix. Then based on Theorem 2.2, the $B_n^{p^*}, n = 2, 3, 4, 5,6$ is an orthogonal matrix.

■

Theorem 3.2 *Let* $B_n^{*(p)}$ *be a matrix that transforms* $vech^*(A)$ *to* $vecp^*(A)$. *Then*

(a) $tr(B_n^{p^*}) = 1$ *for* $n = 2, 3, 4, 5, 6$

(b) $|B_n^{p^*}| = \begin{cases} -1, & if\ n = 2,6 \\ 1, & if\ n = 3,4,5 \end{cases}$

Proof. (a) It is shown that $B_n^{p^*}$ is a matrix in which the element 1 is always located on the main diagonal in the first row and column, while the main diagonal in the other rows and columns contain the element 0. Since a trace is the sum of the entries on the main diagonal, then $tr(B_n^{p^*}) = 1$. (b) Based on Cororllary 3.1 obtained $B_n^{p^*}(B_n^{p^*})^T = (B_n^{p^*})^T B_n^{p^*} = I_{\frac{n(n+1)}{2}}$, so $|B_n^{p^*}(B_n^{p^*})^T| = |I_{\frac{n(n+1)}{2}}|$ or $|B_n^{p^*}||(B_n^{p^*})^T| = |I_{\frac{n(n+1)}{2}}|$. Since $|(B_n^{p^*})^T| = |B_n^{p^*}|$, and we have $|B_n^{p^*}|^2 = |I_{\frac{n(n+1)}{2}}|$. Therefore, $|B_n^{p^*}| = -1$ or $|B_n^{p^*}| = 1$.

Let σ_n be a permutation in $B_n^{p^*}, n = 2, 3, 4, 5, 6$. By using Definition 2.4, we have

- For $n = 2$, $\sigma_2 = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}$,
- For $n = 3$, $\sigma_3 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 2 & 5 & 6 & 3 \end{bmatrix}$
- For $n = 4$, $\sigma_4 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 5 & 2 & 6 & 7 & 3 & 8 & 9 & 10 & 4 \end{bmatrix}$,

- For $n = 5$, $\sigma_5 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 1 & 6 & 2 & 7 & 8 & 3 & 9 & 10 & 11 & 4 & 12 & 13 & 14 & 15 & 5 \end{bmatrix}$, and
- For $n = 6$,

$$\sigma_6 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 \\ 1 & 7 & 2 & 8 & 9 & 3 & 10 & 11 & 12 & 4 & 13 & 14 & 15 & 16 & 5 & 17 & 18 & 19 & 20 & 21 & 6 \end{bmatrix}.$$

Based on Definition 2.5, the number of inversions of σ_2 is 1, σ_3 is 4, σ_4 is 10, σ_5 is 20, and σ_6 is 25. Thus, based on Definitions 2.6 and 2.7, σ_2 and σ_6 are odd permutations, while $\sigma_3, \sigma_4,$ and σ_5 are even permutations so that $\left|B_n^{*(p)}\right| = -1$ for $n = 2{,}6$, and $\left|B_n^{*(p)}\right| = 1$ for $n = 3{,}4{,}5$. The proof is complete.

Tapez une équation ici.

IV. Conclusion

This article provides new operators:i.e $vech^*$ and $vecp^*$. In addition, it provides a definition and finds the properties of a matrix that transforms $vech^*$ and $vecp^*$.

References

[1] H. V. Henderson and S. R. Searle, "Vec and vech operators for matrices, with some uses in Jacobians and multivariate statistics," *The Canadian Journal of Statistics,* vol. 7, no. 1, pp. 65-81, 1979.

[2] K. G. Jinadasa, "Applications of the matrix operators vech and vec," *Linear Algebra and Its Applications,* vol. 101, pp. 73-79, 1988.

[3] T. H. Szatrowski, "Asymptotic nonnull distributions for likelihood ratio statistics in the multivariate normal patterned mean and covariance matrix testing problem," *The Annals of Statistics,* vol. 7, no. 4, pp. 823-837, 1979.

[4] Y. Hardy and W. Steeb, "Vec-operator, Kronecker product and entanglement," *International Journal of Algebra and Computation,* vol. 20, no. 1, pp. 71-76, 2010.

[5] H. Zhang and F. Ding, "On the Kronecker products and their applications," *Journal of Applied Mathematics,* pp. 1-8, 2013.

[6] R. H. Koning, H. Neudecker and T. Wansbeek, "Block Kronecker Products and the vecb Operator," *Linear Algebra and Its Applications,* vol. 149, pp. 165-184, 1991.

[7] I. Ojeda, "Kronecker square roots and the block vec matrix," *The American Mathematical Monthly,* vol. 122, no. 1, pp. 60-64, 2015.

[8] J. R. Schott, Matrix Analysis for Statistics, 3rd ed., New Jersey: John Wiley and Sons, 2017.

[9] D. Nagakura, "On the matrix operator vecp," *Available at SSRN: https://ssrn.com/abstract=2929422 or http://dx.doi.org/10.2139/ssrn.2929422,* pp. 1-12, 2017.

[10] D. Nagakura, "On the relationship between the matrix operator, vech and vecd," *Communication in Statistics-Theory and Method,* vol. 47, no. 13, pp. 3252-3268, 2017.

[11] M. K. Abadir and J. R. Magnus, Matrix Algebra, USA: Cambridge University Press, 2005.

[12] J. A. Gallian, Contemporary Abstract Algebra, 7 ed., Belmon, CA: Brooks/Cole, Cengage Learning, 2010.

[13] R. Piziak and P. L. Odell, Matrix Theory: From Generalized Inverses to Jordan Form, New York: Chapmann & Hall/CRC, 2007.

[14] H. Anton and C. Rorres, Elementary Linear Algebra: Application Version, 5th ed., New Jersey: Wiley, 2004.