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Determination Of The Number Of Arcs Connected To Boundary Disc On A Picture Of Presentation Group

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Abstract – This article discusses the determining of the number of arcs connected to boundary discs on a picture. The picture used is the picture in the presentation of the direct product group $\mathbb{Z}_p \times \mathbb{Z}_q$, $p, q \in \mathbb{Z}^+$. Presentation of the group is $G = \langle a, b;\ a^p, b^q, ab = ba\rangle$. Picture depiction is determined by compiling disc a^p, disc b^q, and disc $ab = ba$, consecutively, and the number disc a^p and disc b^q are unique.

Keywords – picture, arc, disc.

I. Introduction

A *direct product group* is a group resulting from two groups by using the direct product operation. For example A and B is a group, direct product group symbolized by $A \times B$ and is defined as a set $A \times B = \{(a, b) | a \in A,\ b \in B\}$ and operation $(*)$ defined as $(a_1, b_1) * (a_2, b_2) = (a_1a_2, b_1b_2)$. This group has a group presentation. Group presentation represents the elements of each group with one or more generators and relations. Let $G = \langle \boldsymbol{x};\ \boldsymbol{r}\rangle$ be a group presentation that defines the group A, where $\boldsymbol{x}$ is the set of generators and $\boldsymbol{r}$ is the set of relations [1]. The direct product group has a group presentation $G = \langle a, b;\ a^p, b^q, ab = ba\rangle$. Picture is a geometric configuration consisting of disc D^2, disjoint disc Δ, arc and labels [2].

The organization of this paper is as follows: In the Section Basic Theory, we present the definitions of picture and definition of direct product. In the Section Result and Discussion, we present the construction of the picture, and calculation the number of arc that connected to the boundary disc.

II. Basic Theory

In this Section, we present the theory of picture.

Definition 2.1. [2] *A picture Q of P is a geometric configuration consisting of the following:*

a. *A disc D^2 with a basepoint O on the boundary disc D^2, the boundary disc is denoted by ∂D^2.*

b. *Disjoint disc $\Delta_1, \Delta_2, \ldots, \Delta_n$ located in D^2. every disc Δ_i has a base point O_i on $\partial\Delta_i$.*

c. *A number of arc $a_1, a_2, \ldots, a_m$ where each arc located in $D^2 - U_{i=1}^{n}\Delta_i$ and can be a simple closed curve or a simple open curve that connects two points from the boundary disc to the boundary disc, boundary disc to disjoint disc, disjoint disc to disjoint disc. Each arc has an orientation (direction) to the right indicated by the meeting of the short transverse arrow with the arc labeled with the element on the generator.*

d. *If it surrounds $\partial\Delta_i$ once clockwise starting from O_i and reads the label on the arc encountered (if it crosses an arc, say x, with the orientation to the right, then it reads x; otherwise, if it is opposite it reads x^{-1}), then we get a word that belongs to $x \cup x^{-1}$. This word is said to be the label of Δ_i.*

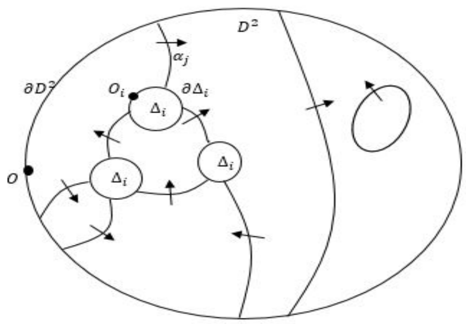

Figure 1. Picture

Following are some of the definitions related to the picture :

1. *A floating circle* is an arc that is closed when it contains no disc.
2. *A floating semi-circle* is an arc that starts and ends at the boundary disc and is not closed.
3. *Inverse pairs* is a spherical picture which has exactly two discs with the same label
4. *Bridge move* is a change of direction and putting two arcs with same label but in different directions.

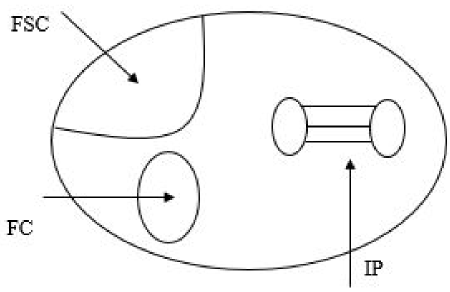

Figure 2. floating circle (FC), floating semi circle (FCS), inverse pairs (IP)

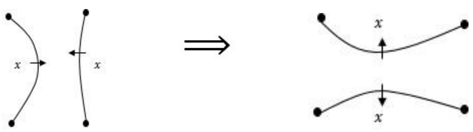

Figure 3. Bridge Move

From Figure 2. and Figure 3. it is explained that if a picture P have exactly in the picture above then it can be said to have floating circle, floating semi circle, inverse pairs, and bridge move.

Definition 2.2. [5] *Let A and B be groups, the direct product $A \times B$ in groups A and B is the set of all ordered pairs $\{(a, b) | a \in A, b \in B\}$ with operation $(*)$ which is defined as $(a_1, b_1) * (a_2, b_2) = (a_1a_2, b_1b_2)$*

Theorem 2.3 [6, 7] *Let A and B be groups defined by presentations $G_1 = \langle \boldsymbol{g}_1; \boldsymbol{s}_1 \rangle$ and $G_2 = \langle \boldsymbol{g}_2; \boldsymbol{s}_2 \rangle$ respectively. Then the presentation $G_1 \times G_2 = \langle \boldsymbol{g}_1, \boldsymbol{g}_2; \boldsymbol{s}_1, \boldsymbol{s}_2, s_1s_2 = s_2s_1\ (s_1 \in \boldsymbol{s}_1, s_2 \in \boldsymbol{s}_2) \rangle$ defines the presentation of direct product group $A \times B$.*

III. RESULT AND DISCUSSION

This paper aims to determine the number of arcs a and arc b connecting to boundary disc in the presentation of direct product $\mathbb{Z}_p \times \mathbb{Z}_q$, $p, q \in \mathbb{Z}^+$. Let $G_1 = \langle a; a^p \rangle$ and $G_2 = \langle b; b^q \rangle$ presentation for $\mathbb{Z}_p$ and $\mathbb{Z}_q$ are respectively, then $G = G_1 \times G_2$ and $G = \langle a, b; a^p, b^q, ab = ba \rangle$. The construction of a picture is limited by one disc a^p and b^q and disc $ab = ba$ are not unique.

Next, the calculation of number arc a and arc b that connected to the boundary disc from picture of presentation of direct product $\mathbb{Z}_p \times \mathbb{Z}_q$ with $p = 2,3$ and $q = 3,4,5$.

1. Picture of direct product $\mathbb{Z}_2 \times \mathbb{Z}_3$ with group presentations $G = \langle a, b;\ a^2, b^3, ab = ba \rangle$.
   a. Picture of $G = \langle a, b;\ a^2, b^3, ab = ba \rangle$ with one disc $ab = ba$ is

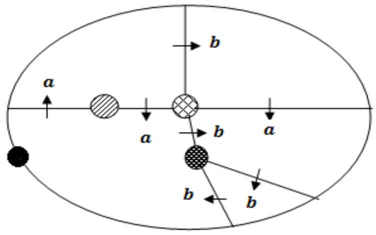

Figure 4. Picture with labels $abab^2$

   b. Picture $G = \langle a, b;\ a^2, b^3, ab = ba \rangle$ with two discs $ab = ba$

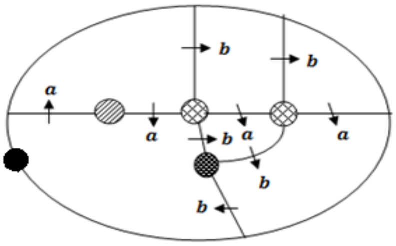

Figure 5. Picture with labels ab^2ab

   c. Picture of $G = \langle a, b;\ a^2, b^3, ab = ba \rangle$ with three discs $aba^{-1}b^{-1}$ is

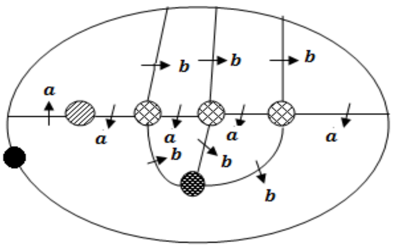

Figure 6. Picture with labels ab^3a

   d. Picture of $G = \langle a, b;\ a^2, b^3, ab = ba \rangle$ with four discs $ab = ba$ is

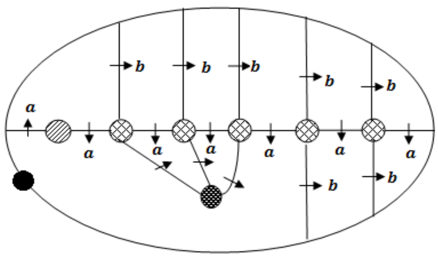

Figure 7. Picture with labels ab^5ab^2

From Figure 4., 5., and 6. in the group presentation $G = \langle a, b;\ a^2, b^3, ab = ba \rangle$ we have two arcs a and three arcs b connected to the boundary disc. From Figure 7., we have two arcs a and seven arcs b connected to boundary disc.

1. Picture of the presentation direct product $\mathbb{Z}^3 \times \mathbb{Z}^4$ with group presentations $G = \langle a, b;\ a^3, b^4, ab = ba \rangle$

   a. Picture of $G = \langle a, b;\ a^3, b^4, ab = ba \rangle$ with one disc $aba^{-1}b^{-1}$

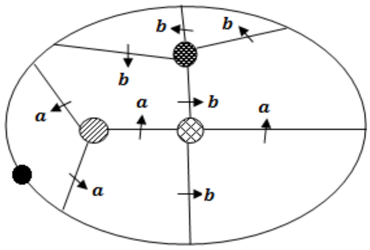

Figure 8. Picture with labels $a^{-1}b^{-1}a^{-1}b^{-1}a^{-1}$

   b. Picture of $G = \langle a, b;\ a^3, b^4, ab = ba \rangle$ with two and three discs $ab = ba$

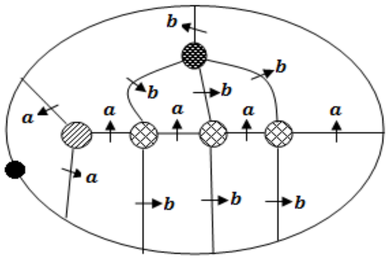

Figure 9. Picture with labels $a^{-1}b^{-1}a^{-1}b^{-3}a^{-1}$

   c. Picture of $G = \langle a, b;\ a^3, b^4, ab = ba \rangle$ with four discs $ab = ba$

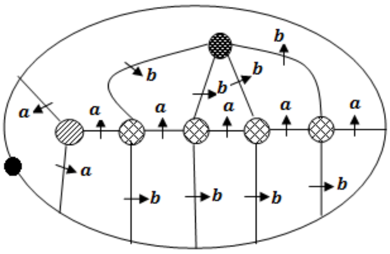

Figure 9. Picture with labels $a^{-1}b^{-4}a^{-1}$

From Figure 7., 8., and 9. in the group presentation $G = \langle a, b;\ a^3, b^4, ab = ba \rangle$ we have three arcs a and four arcs b connected to boundary disc.

2. Picture from the group direct product $\mathrm{Z}^3 \times \mathrm{Z}^5$ with group presentations $\mathrm{G} = \langle \mathrm{a}, \mathrm{b};\ \mathrm{a}^3, \mathrm{b}^5, \mathrm{ab} = \mathrm{ba} \rangle$

   a. Picture $\mathrm{G} = \langle \mathrm{a}, \mathrm{b};\ \mathrm{a}^3, \mathrm{b}^5, \mathrm{ab} = \mathrm{ba} \rangle$ with one disc $ab = ba$

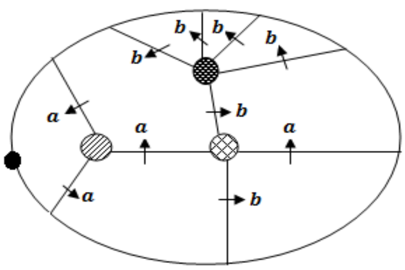

Figure 10. Picture with labels $a^{-1}b^{-4}a^{-1}b^{-1}a^{-1}$

b. Picture $G = \langle a, b;\ a^3, b^5, ab = ba \rangle$ with two discs $ab = ba$

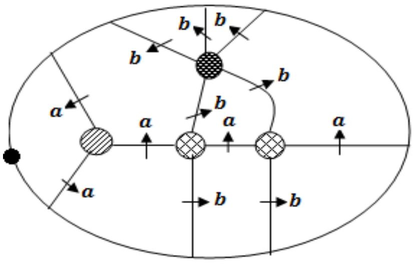

Figure 11. Picture with labels $a^{-1}b^{-3}a^{-1}b^{-2}a^{-1}$

c. Picture $G = \langle a, b;\ a^3, b^5, aba^{-1}b^{-1} \rangle$ with the number disjoint disc $aba^{-1}b^{-1}$ is three and four.

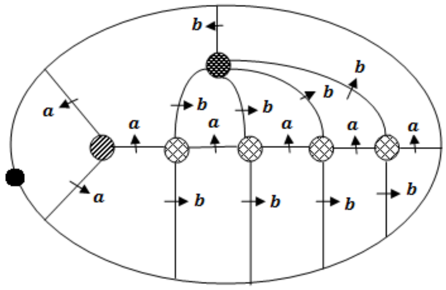

Figure 12. Picture with labels $a^{-1}b^{-1}a^{-1}b^{-4}a^{-1}$

From Figure 10., 11., and 12., in the group presentation $G = \langle a, b;\ a^3, b^5, aba^{-1}b^{-1} \rangle$ we have three arcs a and five arcs b connected to boundary disc.

Proposition. *Let P is picture of $G = \langle a, b;\ a^p, b^q, ab = ba \rangle$, with $p, q \in \mathbb{Z}^+$, then arc a and arc b are connected to the boundary disc for n disc $ab = ba$:*

a. If $n = q$, then there are p arcs a and q arcs b connected to the boundary disc.

b. *If $n > q$, then there are p arcs a and $2n - q$ arcs b connected to the boundary disc.*

Proof:

a. For $n = q$, we have

- The arc a that is connected to the boundary disc are p. Let P be a picture of $G = \langle a, b;\ a^p, b^q, aba^{-1}b^{-1} \rangle$, with $p, q \in \mathbb{Z}^+$. Since arc a is in $ab = ba$ and disc a^p is in the first position in the picture, then there is an arc a that is connected to the disc $ab = ba$ and the number of arcs a connected to the boundary disc are $p - 1$. Next, since b^q and $n = q$ then all arc b is connected to $ab = ba$, while there is one arc a on disc $ab = ba$ in the last position connected to the boundary disc. So, the number of arcs a connected to the boundary disc are $p - 1 + 1 = p$.
- The arc b connected to the boundary disc is q. Therefore, since disc b^q is unique and arc b is in $ab = ba$, so there is arc b connected to disc $ab = ba$. If number of disc $ab = ba$ are n with $n = q$, then the whole arc b in disc b^q connected to disc $ab = ba$. Meanwhile, in disc $ab = ba$ there are arc b connected to the boundary disc.

b. For $n > q$, we have

- The arc a that is connected to the boundary disc are p. Let P be a picture of $G = \langle a, b;\ a^p, b^q, ab = ba \rangle$, with $p, q \in \mathbb{Z}^+$. Since arc a is in $ab = ba$ and disc a^p is in the first position in the picture, then there is an arc a that is connected to the disc $ab = ba$ and the number of arcs a connected to the boundary disc are $p - 1$. Next, since b^q and $n = q$ then all arc b is connected to $ab = ba$, while there is one arc a on disc $ab = ba$ in the last position connected to the boundary disc. So the number of arcs a connected to the boundary disc are $p - 1 + 1 = p$

- The arc b connected to boundary disc is $2n - q$. Let Q picture of $G = \langle a, b;\ a^p, b^q, ab = ba \rangle$, with $p, q \in \mathbb{Z}^+$, Therefore, arc b^q is unique and for $n = q$, then the whole arcs b connected to disc $ab = ba$ and the number of arc b connected to the boundary disc are q. Therefore, if we add the disc $ab = ba$ until $n > q$, then there is arc b as much as $2n - q$.

IV. Conclusion

Given a direct product group $\mathbb{Z}_{\boldsymbol{p}} \times \mathbb{Z}_{\boldsymbol{q}}$, $p, q \in \mathbb{Z}^+$ with group presentations $G = \langle a, b;\ a^p, b^q, ab = ba \rangle$and we can construct a picture from this. We have the number of arcs connected to boundary disc that can be seen from label in the picture.

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