International Journal of Progressive Sciences and Technologies (IJPSAT)
ISSN: 2509-0119.

Vol. 36 No. 1 December 2022, pp. 198-203

On the Ramsey Minimal Graphs for Matching and Disjoint Union of Complete Bipartite Graphs

Muhammad Rafif Fajri, Lyra Yulianti*, Des Welyyanti

Department of Mathematics and Data Science,

Faculty of Mathematics and Natural Sciences, Andalas University

Kampus UNAND Limau Manis Padang-25163, Indonesia

Abstract – Let G and H be two arbitrary graphs. The notation $F \rightarrow (G, H)$ means that any red-blue coloring of every edge of graph F always resulting a red subgraph containing G or a blue subgraph containing H. Denote $F^* := F\backslash\{e\}$ for any edge of F. The notation $F^* \nrightarrow (G, H)$ means that there exists a coloring of F^* such that F^* does not contain red G and blue H. The class $\mathcal{R}(G, H)$ states a set of graphs satisfying: (1) $F \rightarrow (G, H)$. (2) $\forall e \in F$, $F^* := F\backslash\{e\}$, $F^* \nrightarrow (G, H)$. In this paper, some graphs in $\mathcal{R}(aK_2, bK_{3,n})$ are obtained, where aK_2 is a matching and $bK_{3,n}$ is a disjoint union of complete bipartite graphs $K_{3,n}$ for positive integer n.

Keywords – Ramsey minimal graph, Matching, Complete bipartite graph

I. INTRODUCTION

All graphs in this paper are considered undirected, finite and simple. Let G and H be two arbitrary graphs. If the edges of G are given arbitrary red-blue coloring, then the notation $F \rightarrow (G, H)$ means that F contains red subgraph G or blue subgraph H. If the coloring makes F does not contain red G and blue H, then we denote that $F \nrightarrow (G, H)$. Graph F is a Ramsey (G, H)-minimal graph, denoted by $F \in \mathcal{R}(G, H)$, if $F \rightarrow (G, H)$ and $F\backslash\{e\} \nrightarrow (G, H)$ $\forall e \in E(F)$ [3]. Other notations and definitions are taken from Diestel [7].

Burr et al. [4] showed that for every positive integer m and an arbitrary graph H, the class $\mathcal{R}(mK_2, H)$ is finite. Some results related to the finite class are as follows. Burr et al. [5] discussed about the Ramsey minimal graphs for $\mathcal{R}(2K_2, H)$, where H is a matching. Baskoro and Wijaya [1] determined some graphs in $\mathcal{R}(2K_2, K_4)$, where K_4 is a complete graph on 4 vertices. Baskoro and Yulianti [2] focused on the graphs in $\mathcal{R}(2K_2, P_n)$, where P_n is a path on n vertices.

In [10] Mengersen and Oeckermann considered about $\mathcal{R}(2K_2, K_{1,n})$, where $K_{1,n}$ is a star on $n + 1$ vertices. Next, Muhshi and Baskoro determined the graphs in $\mathcal{R}(3K_2, P_3)$. Wijaya et al. [14] listed some graphs in $\mathcal{R}(3K_2, K_3)$. Wijaya et al. also gave complete list of graphs in $\mathcal{R}(2K_2, K_4)$ and $\mathcal{R}(2K_2, C_4)$ (see [19] and [20]). Moreover, Wijaya et al. [15] discussed about the characterizations of graphs in $\mathcal{R}(mK_2, H)$ for an arbitrary graph H. Wijaya et al. [13] also gave some characterizations of graphs in $\mathcal{R}(2K_2, 2H)$ for an arbitrary graph H. Another results are graphs in $\mathcal{R}(4K_2, P_3)$, $\mathcal{R}(mK_2, P_3)$ and $\mathcal{R}(4K_2, P_3)$ (see [16], [17], and [18]).

Nabila et al. [12] gave some graphs in $\mathcal{R}(aK_2, bK_{1,n})$ and Fajri et al. [8] gave some graph in $\mathcal{R}(aK_2, bK_{2,n})$. In this paper we study the finite class $\mathcal{R}(aK_2, bK_{3,n})$, where aK_2 is a union of complete graphs K_2 and $bK_{3,n}$ is a union of complete bipartite graphs $K_{3,n}$ for positive integer n.

II. Main Result

The definition of $\Omega(t,3,n)$, for $t,n \in \mathbb{N}$ is given in Definition 2.

Definition 2,1. Let t and n be two positive integers. The vertex set and edge set of $\Omega(t,3,n)$ are given as follows.

$$V(\Omega(t,3,n)) = \{x_i, y_j, z_k | 1 \le i \le 3; 1 \le j \le n+t; 1 \le k \le t\},$$

$$E(\Omega(t,3,n)) = \{x_i y_j, y_j z_k | 1 \le i \le 3; 1 \le k \le t \text{ for } 1 \le j \le n; j-n+1 \le k \le t \text{ for } n+1 \le j \le n+t\}.$$

Graph $\Omega(t,3,n)$ for $t,n \in \mathbb{N}$ is given in Figure 1. It can easily be seen that the graph is a non-complete bipartite graph.

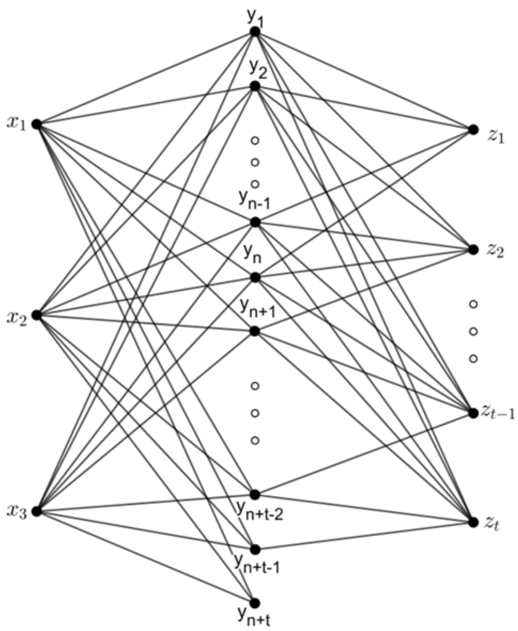

Figure 1: $\Omega(t,3,n)$

In Lemma 2.1 - Lemma 2.2, we give the properties of $\Omega(t,3,n)$ for $t,n \in \mathbb{N}$. Let $\Omega(t,3,n)$ be a graph in Definition 2.

Lemma 2.2. The graph $\Omega(t,3,n)$ contains a perfect matching if $n = 3$, contains a non-perfect matching if $n \ne 3$, and the maximum cardinality of the matching is $|M(\Omega(t,3,n))| = 3 + t$, for $t,n \in \mathbb{N}$.

Proof. Partition the set $V(\Omega(t,3,n))$ into two partitions, namely $V(U) = \{x_i \in V(\Omega(t,3,n)) | 1 \le i \le 3\} \cup \{z_k \in V(\Omega(t,3,n)) | 1 \le k \le t\}$ and $V(W) = \{y_j \in V(\Omega(t,3,n)) | 1 \le j \le n+t\}$. Note that the number of vertices in the first and second partition sets are, respectively, $|V(U)| = 3 + t$ and $|V(W)| = n + t$. The cardinality of the maximum matching of $\Omega(t,3,n)$ is $|M(\Omega(t,3,n)| = 3 + t$, with $M(\Omega(t,3,n) = \{x_i y_{i+t}, y_j z_j \in E(\Omega(t,3,n)) | 1 \le i \le 3; 1 \le j \le t; \}$. Next, if $n = 3$, then $M_v(\Omega(t,3,n)) = M(\Omega(t,3,n))$ so that the graph $\Omega(t,3,n)$ have a perfect matching. Furthermore, if $m \ne n$, then there is a vertex set $N = \{y_i \in V(\Omega(t,m,n)) | 3 + t + 1 \le i \le n + t\}$ such that $N \not\subseteq M_V(\Omega(t,3,n))$ such that the graph $\Omega(t,3,n)$ is not a perfect matching.

(Q.E.D)

Lemma **2.3.** Let $\Omega(t,3,n)$ be a graph in Definition 2. Let $F \subseteq \Omega(t,3,n)$ with $|M(F)| = k$. Let $H = \bigcup_{i=1}^{k} K_{1,s(i)} \subseteq \Omega(t,3,n)$, where $s(i)$ is the maximum degree that can be formed from the i^{th} star graph. Then, $F \subseteq H$.

Proof. Because $\Omega(t,3,n)$ is bipartite, and $F, H \subseteq \Omega(t,3,n)$, then F and H are also bipartite. Therefore, there is no odd cycle and no complete graph K_n for $n \ge 3$ in graph F and H. By Lemma 2.2, since $|M(\Omega(t,3,n))| = 3 + t$ and $F \subseteq \Omega(t,3,n)$, then $|M(F)| = k$, for $1 \le k \le 3 + t$. Next, we construct $H = \bigcup_{i=1}^{k} K_{1,s(i)} \subseteq \Omega(t,3,n)$, with $s(i)$ is the maximum degree that can be formed from the i^{th} star graph. Since every vertex on graph H has a maximum degree respect to $\Omega(t,3,n)$, so the number of vertices and edges of graph H is also maximum. Since $|M(H)| = |M(F)| = k$, the graphs $H, F \subseteq \Omega(t, 3, n)$, and H are graphs with a maximum number of vertices and edges, then $F \subseteq H$.

(Q.E.D)

In Theorem 2.4, we show that for some positive integers t and n, $\Omega(t, 3, n)$ is a Ramsey-minimal graph for $((t+1)K_2, K_{3,n})$.

Theorem 2.4. Let $\Omega(t, 3, n)$ be a graph in Definition 2. Let t and n be two positive integers. Then, $\Omega(t, 3, n) \in \mathcal{R}((t+1)K_2, K_{3,n})$.

Proof. First, we show that $\Omega(t, 3, n) \to ((t+1)K_2, K_{3,n})$. Consider any red-blue coloring of the edges of the graph $\Omega(t, 3, n)$. Suppose that there is no red $(t+1)K_2$ in the coloring. Therefore, the possible maximum red subgraph is tK_2. The graphs that may contain red tK_2 are complete graphs K_{2t+1}, odd cycle C_{2t+1}, path P_{2t+1}, and any other graph that has t as the cardinality of their maximum matching. Since $\Omega(t, 3, n)$ is a bipartite graph, we know that there is no odd cycle in the graph. Therefore, the possibility of a red graph in the form of C_{2t+1}, or a combination of several odd-cycle graphs with a maximum cardinality of matching t, can be ignored. Next, since $C_3 \subseteq \Omega(t, 3, n)$ and $C_3 \subseteq K_t$ for $t \geq 3$, we know that there is no complete graph K_t in the graph $\Omega(t, m, n)$. Therefore, the possibility of a red graph in the form of K_{2t+1}, or a combination of several complete graphs K_s, for $s \geq 3$ with a maximum cardinality of matching t, can also be ignored.

Denote $\mathbb{F}$ **as** the set containing all graphs with the cardinality of the maximum matching of t and $F \subseteq \Omega(t, 3, n), \forall F \in \mathbb{F}$. It can be seen that $|M(tK_2)| = |M(F)| = t, \forall F \in \mathbb{F}$. From Lemma 2.3, we know that $F \subseteq \bigcup_{i=1}^{t} K_{1,s(i)} = H \subseteq \Omega(t, 3, n)$, where $s(i)$ is the maximum degree that can be formed from the i^{th} star graph in $\Omega(t, 3, n)$. Therefore, the combination of t star graphs has represented all cases of the pissibilities of the red tK_2 in $\Omega(t, 3, n)$.

We **construct** the red-blue coloring of $\Omega(t, 3, n)$ as follows.

1. Every edge that incident with r vertices on $X = \{x_i \in V(\Omega(t, 3, n)) | 1 \leq i \leq 3\}$, with $0 \leq r \leq t$ are colored red. Denote this set of red vertices as R.

2. Every edge that incident with s vertices on $Y = \{y_j \in V(\Omega(t, 3, n)) | 1 \leq j \leq n+t\}$, with $0 \leq s \leq t-r$ are colored red. Denote this set of red vertices as S.

3. Every edge that incident with $t-r-s$ vertices on $Z = \{z_k \in V(\Omega(t, 3, n)) | 1 \leq k \leq t\}$ are colored red. Denote this set of red vertices as P.

4. The remaining edge are colored blue. Denote this blue subgraph as B.

Consider the vertex set $V(B)$. Denote the vertex sets $B_X = (V(B) \cap X) - R$, $B_Y = (V(B) \cap Y) - S$, and $B_Z = (V(B) \cap Z) - P$. Take all the vertices on B_X, n the first point on B_Y, and r the last vertices on B_Z. Denote the set of all the vertices that have been taken as $\mathbb{B}_V$. Then, add some edges between every vertex in $\mathbb{B}_V$, and denote $\mathbb{B}_E$ as the set containing these new edges, with the condition $\mathbb{B}_E \subset E(B)$. Note that the vertex set $\mathbb{B}_V$ and the edge set $\mathbb{B}_E$ build up the graph $K_{3,n}$. Then, for every possibility of red tK_2, we always have a blue $K_{3,n}$. Therefore, $\Omega(t, n, m) \to ((t+1)K_2, K_{3,n})$.

Next, we show that $\forall e \in E(\Omega(t, 3, n))$, $\Omega(t, 3, n)^* := \Omega(t, 3, n) \setminus \{e\} \nrightarrow ((t+1)K_2, K_{3,n})$. We list all the possibilities of the red-blue coloring of the edges of $\Omega(t, 3, n)^*$ such that it does not contain $(p+1)K_2$ red and $K_{3,n}$ blue in Table 1 as follows.

Table 1: The Possibilities of the red-blue coloring of the edges of $\Omega(t,3,n)^*$

such that it does not contain red $(p+1)K_2$ and blue $K_{3,n}$

Case	Edge deletion	For	Condition
1	x_iy_j	$1 \le i \le 3; 1 \le j \le n$	$t < 3; i < t+2$
	Red Edge Incident with	x_k	$1 \le k \le t+1; k \ne i$
2	x_iy_j	$1 \le i \le 3; 1 \le j \le n$	$t < 3; i \ge t+2.$
	Red Edge Incident with	x_k	$1 \le k \le p$
3	x_iy_j	$1 \le i \le 3; n+1 \le j \le n+t-1$	$t < 3; i < t+n+2-j.$
	Red Edge Incident with	x_k	$1 \le k \le t+n+1-j; k \ne i$
		y_r	$1 \le r \le j-n$
4	x_iy_j	$1 \le i \le 3; n+1 \le j \le n+t-1$	$t < 3; i \ge t+n+2-j.$
	Red Edge Incident with	x_k	$1 \le k \le t+n-j$
		y_r	$1 \le r \le j-n$
5	x_iy_{n+t}	$1 \le i \le 3$	$t < 3.$
	Red Edge Incident with	y_r	$1 \le r \le t$
6	x_iy_j	$1 \le i \le 3; 1 \le j \le n$	$t \ge 3.$
	Red Edge Incident with	x_k	$1 \le k \le 3; k \ne i$
		z_s	$t-3+1 \le s \le t$
7	x_iy_j	$1 \le i \le 3; n+1 \le j \le n+t-3$	$t \ge 3.$
	Red Edge Incident with	x_k	$1 \le k \le 3; k \ne i$
		y_r	$1 \le r \le j-n$
		z_s	$j-n+3 \le s \le t$
8	x_iy_j	$1 \le i \le 3; n+t-2 \le j \le n+t-1$	$t \ge 3; i < j-n-t+3.$
	Red Edge Incident with	x_k	$j-n-t+4 \le k \le 3$
		y_r	$1 \le r \le j-n$
9	x_iy_j	$1 \le i \le 3; n+t-2 \le j \le n+t-1$	$t \ge 3; i \ge j-n-t+3.$
	Red Edge Incident with	x_k	$j-n-t+3 \le k \le 3; k \ne i$
		y_r	$1 \le r \le j-n$
10	x_iy_{n+t}	$1 \le i \le 3$	$t \ge 3$
	Red Edge Incident with	x_k	$1 \le k \le t$
11	y_iz_1	$1 \le i \le n$	
	Red Edge Incident with	x_k	$k = 1$
		z_s	$2 \le s \le t$
12	y_iz_j	$1 \le i \le n+j-1; 2 \le j \le t-1$	$i < j$
	Red Edge Incident with	x_k	$k = 1$
		y_r	$1 \le r \le j; r \ne i$
		z_s	$j+1 \le s \le t$
13	y_iz_j	$1 \le i \le n+j-1; 2 \le j \le t-1$	$i \ge j$
	Red Edge Incident with	x_k	$k = 1$
		y_r	$1 \le r \le j-1$

		z_s	$j+1 \le r \le t-1$
14	$y_i z_t$	$1 \le i \le n+t-1$	$i \ge j$
	Red Edge Incident with	x_k	$k = 1$
		y_r	$1 \le r \le t;\ r \ne i$

For **example**, consider Case 1. This case holds for $t < 3$. One edge that is deleted in the graph $\Omega(t,3,n)^*$ is one of $x_i y_j$, for $1 \le i \le 3$ and $1 \le j \le n$. If $i < t+2$, then color all edges that incident to x_k, for $1 \le k \le t+1$ and $k \ne i$, with red color. The remaining edges are colored blue. Note that there is neither red nor blue $(t+1)K_2$ in the red-blue $\Omega(t,3,n)^*$ coloring. Other cases are explained similarly. Based on the 14 cases above, we have that $\Omega(t,3,n)^* \nrightarrow ((t+1)K_2, K_{3,n})$.

(Q.E.D)

In **Definition** 2.5, we define graph $(a+b-1)K_{3,n}$ for $n \in \mathbb{N}$.

Definition 2.5. Let a, b, and n be three positive integers. Let $K_{3,n}^{(s)}$ be the s^{th} complete bipartite graph, for $1 \le s \le a+b-1$. Denote $(a+b-1)K_{3,n} = \cup_{s=1}^{a+b-1} K_{3,n}^{(s)}$. The vertex set and edge set of $(a+b-1)K_{3,n}$ are given as follows.

$$V(K_{3,n}^{(t)}) = \{x_{t,i}, y_{t,j} | 1 \le i \le 3; 1 \le j \le n; 1 \le t \le a+b-1\},$$

$$E(K_{3,n}^{(t)}) = \{x_{t,i} y_{t,j} | 1 \le i \le 3; 1 \le j \le n; 1 \le t \le a+b-1\}.$$

Graph $(a+b-1)K_{3,n}$ is given in Figure 2.

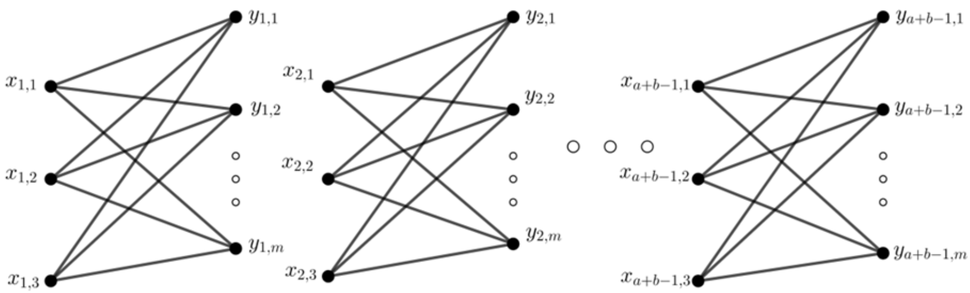

Figure 2: $(a+b-1)K_{3,n}$

In **Theorem** 2.6, we show that for some positive integers a, b, and n, the graph $(a+b-1)K_{3,n}$ is a Ramsey-minimal graph for $(aK_2, bK_{3,n})$.

Theorem 2.6. Let a, b, and n be three positive integers. Let $(a+b-1)K_{3,n}$ be a graph in Definition 0. Then, $(a+b-1)K_{3,n} \in \mathcal{R}(aK_2, bK_{3,n})$.

Proof. First, we show that $(a+b-1)K_{m,n} \to (aK_2, bK_{3,n})$. Consider any red-blue coloring of the edges of the graph $(a+b-1)K_{3,n}$. Suppose that there is no red aK_2 in the coloring. Therefore, the possible maximum red subgraph is $(a-1)K_2$. Without loss of generality, color any edge of the graph $K_{3,n}^{(i)}$, for $1 \le i \le a-1$ with one red K_2 each, and the remaining edge are colored blue. Note that the subgraph $K_{3,n}^{(i)}$ does not contain $K_{3,n}$ blue and b subgraph $K_{3,n}^{(j)}$ contain $bK_{3,n}$ blue, for $a+1 \le j \le a+b-1$. Therefore, $(a+b-1)K_{3,n} \to (aK_2, bK_{3,n})$.

Next, we show that $\forall e \in (a+b-1)K_{3,n}$, $(a+b-1)K_{3,n}^* := (a+b-1)K_{3,n} \backslash \{e\} \nrightarrow (aK_2, bK_{3,n})$. Without loss of generality, let the deleted edge is in the subgraph $K_{3,n}^{(1)}$. Then, color any edge of the subgraph $K_{3,n}^{(i)}$, for $2 \le i \le a$ with one red K_2, and the remaining edge are colored blue. Note that the subgraph $K_{3,n}^{(i)}$ does not contain $K_{3,n}$ blue and b subgraph $K_{3,n}^{(j)}$ only contains $(b-1)K_{3,n}$ blue, for $a+1 \le j \le a+b-1$. Therefore, $(a+b-1)K_{3,n} \nrightarrow (aK_2, bK_{3,n})$.

(Q.E.D)

III. Conclusions

In this **paper**, we have determined that $\Omega(t, 3, n) \in \mathcal{R}((t + 1)K_2, K_{3,n})$ and $(a + b - 1)K_{3,n} \in \mathcal{R}(aK_2, bK_{3,n})$.

References

[1] Baskoro, E.T., Wijaya, K. 2015. On Ramsey $(2K_2, K_4)$-minimal graphs. In: *Mathematics in the 21st Century, Springer Proceedings in Mathematics & Statistics* Vol. **98**: 11 – 17

[2] Baskoro, E.T., Yulianti, L. 2011. On Ramsey minimal graphs for $2K_2$ versus P_n. *Adv. Appl. Discrete Math.* **8**(2): 83 – 90

[3] Burr, S. A., Erdős, P., Lovasz, L. 1976. On Graph of Ramsey Type, *Ars Combinatoria*: 167 – 190.

[4] Burr, S.A., Erdös, P., Faudree, R.J., Schelp, R.H. 1978. A class of Ramsey-finite graphs. In: *Proceeding of the Ninth Southeastern Conference on Combinatorics, Graph Theory and Computing*: 171 – 180

[5] Burr, S.A., Erdös, P., Faudree, R.J., Schelp, R.H.: 1981. Ramsey minimal graphs for matchings. In: *The Theory and Applications of Graphs* (Kalamazoo, Mich.): 159 – 168

[6] Burr, S.A., Erdös, P., Faudree, R.J., Rousseau, C.C., Schelp, R.H. 1982. Ramsey minimal graphs for forests. *Discrete Math.* **38**(1): 23 – 32

[7] Diestel, R., 2017, *Graph Theory*, 5^{th} ed., Springer-Verlag New York Inc., New York

[8] Fajri, M. R., Yulianti, L., Welyyanti, D., 2022. Ramsey Minimal Graph for Union of Complete Graph and Complete Bipartite Graph, submitted to *Electronic Journal of Graph Theory and Applications*

[9] Fitriani D., Salman A.N.M. 2016. Rainbow connection number of amalgamation of some graph. *AKCE Internasional Journal of Graph and Combinatorics* **13**: 90 –99

[10] Mengersen, I., Oeckermann, J. 1999. Matching-star Ramsey sets. *Discrete Appl. Math.* **95**: 417 – 424

[11] Muhshi, H., Baskoro, E.T. 2012. On Ramsey $(3K_2, P_3)$-minimal graphs. *AIP Conference Proceeding* **1450**: 110 – 117

[12] Nabila, M., Yulianti, L., Syafwan, M., 2019. Graf Ramsey Minimal Untuk Pasangan Gabungan Graf Lengkap K_2 dan Gabungan Graf Bintang $K_{1,n}$ (in Bahasa), *Jurnal Matematika UNAND*. **8**(1): 195 – 200

[13] Wijaya, K. , Baskoro, E. T. 2016. On Ramsey $(2K_2, 2H)$-Minimal Graphs, *Springer India, J. Math*, **186**: 219 – 223

[14] Wijaya, K., Baskoro, E.T., Assiyatun, H., Suprijanto, D. 2016. On Ramsey $(3K_2, K_3)$-minimal graphs. In: *AIP Conference Proceedings* Vol. **1707**, pp. 020025. doi:10.1063/1.4940826

[15] Wijaya, K., Baskoro, E.T., Assiyatun, H., Suprijanto, D. 2016. On Ramsey (mK_2, H)-Minimal Graphs, *Graphs and Combinatorics* Vol. **23**: 233 – 243

[16] Wijaya, K., Baskoro, E.T., Assiyatun, H., Suprijanto, D. 2018, On Ramsey $(4K_2, P_3)$-minimal graphs, *AKCE International Journal of Graphs and Combinatorics* **15**(2): 174 – 186

[17] Wijaya, K., Baskoro, E.T., Assiyatun, H., Suprijanto, D. 2015. On unicyclic Ramsey (mK_2, P_3)-minimal graphs. *Proc. Comput. Sci*. **74**: 10 – 14

[18] Wijaya, K., Baskoro, E.T., Assiyatun, H., Suprijanto, D., 2020, Subdivision of graphs in $R(mK_2, P_4)$, *Heliyon* **6**(6): e03843

[19] Wijaya, K., Baskoro, E.T., Assiyatun, H., Suprijanto, D. 2015. The complete list of Ramsey $(2K_2, K_4)$-minimal graphs. *Electron. J. Graph Theory Appl.* **3**(2) : 216 – 227

[20] Wijaya, K., Yulianti, L., Baskoro, E.T., Assiyatun, H., Suprijanto, D. 2015. All Ramsey $(2K_2, C_4)$-minimal graphs. *J. Algorithms Comput.* **46**: 9 – 25