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Application of Toeplitz Effects to Hardy Space

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Abstract – We're reviewing the relationship between Fourier Analytics and Hardy Spaces $H^{H^{\alpha+1}}$. Leads to stable vertebral levels of bilateral and unilateral transformations and studying Toeplitz and metric effects. That Gives elementary spectroscopic theories of the Toeplitz effects. And the Toeplitz algebraic effect.

Keywords – Hardy Spaces, Toeplitz, effects, metric, algebraic, Hilbert.

1- A Hardy space $H^{\alpha+1}$. Put $\mathbb{T}$ Be the circle group and give it $\mathbb{T}$ with a natural arc to measure the height, symbolized by it $d\lambda$. We are writing $L^{\alpha+1}(\mathbb{T})$ by $L^{\alpha+1}(\mathbb{T}, d\lambda)$. Then $d\lambda(\mathbb{T}) < \infty$, $L^{\varepsilon+1}(\mathbb{T}) \subset L^{\alpha+1}(\mathbb{T})$. Where $0 \leq \alpha \leq \varepsilon$, can be

$$\int f(\lambda)d\lambda = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})dt \tag{1}$$

For every $n \in \mathbb{Z}$, The continuous function can be specified $\mathcal{E}_n$ can be

$$\mathcal{E}_n: \mathbb{T} \to \mathbb{T}, \lambda \longmapsto \lambda^n. \tag{2}$$

We refer to Γ and Γ_+ as linear space of the sets $\{\mathcal{E}_n | n \in \mathbb{Z}\}$ and $\{\mathcal{E}_n | n \in \mathbb{N}\}$ Successively. The elements in Γ and Γ_+ for trigonometric polynomials and also for analytical trigonometric polynomials successively.

Lemma 1.1.

(1) Γ is a $*$- subalgebra at $C(\mathbb{T})$

(2) To $0 \leq \alpha \leq +\infty$, the Γ its $L^{\alpha+1}$-anorm dense at $L^{\alpha+1}(\mathbb{T})$.

(3) $(\varepsilon_n)_{n\in\mathbb{Z}}$ is the orthogonal basic of the Hilbert space $L^2(\mathbb{T})$.

Proof: (1) it's obvious. From the Stone-Weierstrass notion, Γ it's anorm dense at $C(\mathbb{T})$, from at $C(\mathbb{T})$ it's $L^{\alpha+1}$-anorm dense at $L^{\alpha+1}(\mathbb{T})$, form (1) can get (2) and (3) Comes directly from (2).

Definition 1.2. When $f \in L^1(\mathbb{T}), n \in \mathbb{Z}$, then $n - th$ Fourier modulus of f is defined as

$$\hat{f}(n) = \int f(\lambda)\overline{\varepsilon_n(\lambda)}d\lambda$$

from a function

$$\hat{f}: \mathbb{Z} \to \mathbb{C},\ n \longmapsto \hat{f}(n)$$

it means Fourier transformation f.

Definition 1.3 Put $0 \leq \alpha \leq +\infty$. Know define the Hardy space $H^{\alpha+1}$it can

$$H^{\alpha+1} = \{f \in L^{\alpha+1}(\mathbb{T}) \big| \hat{f}(n) = 0, (n < 0)\} \qquad (3)$$

$H^{\alpha+1}$ its $L^{\alpha+1}$-anormed locked a subspace vector to $L^{\alpha+1}(\mathbb{T})$. Particularly, H^2 donated that L^2 included product is a Hilbert space it an orthonormaed basic Γ_+.

The Hardy spaces contain an explanation of the analytical functions on the open unit disk $\mathbb{D} \subset \mathbb{C}$. To meet the characteristics of a close-edged twist $\partial\mathbb{D} = \mathbb{T}$. Let us can find out when $\alpha = 1$. Let $f: \mathbb{D} \to \mathbb{C}$ it is an analytical functions, $0 \leq r < 1$, that is adjust $f_r(\lambda) := f(r\lambda), (\lambda \in \mathbb{T})$, and $\|f\|_{H^2(\mathbb{D})} \coloneqq sup_{0<r<1}\|f_r\|_{L^2(\mathbb{T})}$. The Hardy space can be defined by $H^2(\mathbb{D})$ to the unit disk at

$$H^2(\mathbb{D}) = \{f: \mathbb{D} \to \mathbb{C} | f \text{ } its\ analytic\ on\ \mathbb{D}\ and\ \|f\|_{H^2(\mathbb{D})} < +\infty\}.$$

Whether $Z \in \mathbb{D}$, it defined that $T_Z: H^1 \to \mathbb{C}$, in a way of

$$T_Z(f) = \int \frac{f(\lambda)}{1 - z\bar{\lambda}} d\lambda,\ \ (f \in H^1).$$

When $\frac{1}{1-z\bar{\lambda}} = \sum_{n=0}^{\infty} Z^n \lambda^{-n}$, we see that $T_Z(f) = \sum_{n=0}^{\infty} \hat{f}(n) Z^n$. With a direct computation, T_Z its specified linear function at H^1, then a functional

$$\tilde{f}: \mathbb{D} \to \mathbb{C}, z \mapsto T_Z(f)$$

its analytical. Furthermore, at every $Z \in \mathbb{D}$ & $f, g \in H^2$, $T_Z(fg) = T_Z(f)T_Z(g)$.

There is a relation between H^2 & $H^2(\mathbb{D})$, expound in the following theory.

Fact 1.4. (1) $f \in H^2(\mathbb{D})$ if there is a sequence $(a_n)_{n\in\mathbb{N}} \in \ell^2$, so $f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n$, $(\lambda \in \mathbb{D})$[2]. Furthermore, $\|f\|_{H^2(\mathbb{D})} = (\sum_{n=0}^{\infty} |a_n|^2)^{\frac{1}{2}}$

(2) $H^2(\mathbb{D})$ its Hilbert space with numerical multiplication property

$$\langle f, g \rangle_{H^2(\mathbb{D})} = \sum_{n=0}^{\infty} a_n \overline{b_n}$$

since $a_n = f^{(n)}(0)/n!$ & $b_n = g^{(n)}(0)/n!$

(3) Amap

$$H^2 \to H^2(\mathbb{D}), f \mapsto \tilde{f},$$

it's the only factor.

Then it gets us thinking about functions of H^2 they are the bounded a value of the functions $H^2(\mathbb{D})$, is method of boundary factors at (3) in fact (1.4)[1].

These examples illustrate numerical examples of the diversity of analytical behavior.

Lemma 1.5. Let $f, \bar{f} \in H^1$, where can $\alpha \in \mathbb{C}$, so $f = \alpha$.

Proof: I think that $f = \bar{f}$ & whether $\alpha = \int f(\lambda) d\lambda \in \mathbb{C}$, so that

$$\bar{\alpha} = \int \overline{f(\lambda)} d\lambda = \int f(\lambda) d\lambda = \alpha,$$

then $\alpha \in \mathbb{R}$. When $n < 0$, so that

$$\left(f \widehat{- \alpha\varepsilon_0}\right)(n) = \int (f(\lambda) - \alpha)\overline{\varepsilon_n(\lambda)}d\lambda = 0 - \alpha 0 = 0,$$

As well as

$$(f \widehat{- \alpha\varepsilon_0})(n) = \int \left((f(\lambda) - \alpha)\overline{\varepsilon_n(\lambda)}\right) d\lambda = \overline{\int \left((f(\lambda) - \alpha)\overline{\varepsilon_{-n}(\lambda)}\right) d\lambda} = 0,$$

When α in fact that $f = \bar{f}$, in the last,

$$\left(f \widehat{- \alpha\varepsilon_0}\right)(0) = \int (f(\lambda) - \alpha)d\lambda = \alpha - \alpha = 0,$$

so that $f = \alpha$.

Now we just assume that $f, \bar{f} \in H^1$, so $\mathrm{Re}(f)$ & $\mathrm{Im}(f)$, it is of real value at H^1, so we find the solutions to these for those functions first, so that f is constant.

Sub-fixed spaces for transformations in bilateral and unilateral sides.

First, we return to the relationship between multiplication and the binary transformation of the base (ε_n), and determine the invariant subspaces for it.

We put $(\varphi_{\varepsilon+1}) \in L^\infty(\mathbb{T})$, the multiplication effect $M_{(\varphi_{\varepsilon+1})} \in B\left(L^2(\mathbb{T})\right)$, with $(\varphi_{\varepsilon+1})$, it is define with,

$$M_{(\varphi_{\varepsilon+1})}(f) = (\varphi_{\varepsilon+1})f,\ \left(f \in L^2(\mathbb{T})\right).$$

A map

$$M_*: L^\infty(\mathbb{T}) \to B\left(L^2(\mathbb{T})\right),\ (\varphi_{\varepsilon+1}) \longmapsto M_{(\varphi_{\varepsilon+1})}$$

is an isometric $*$-homomorphism.

Lemma 1.6. (1) H^∞means finite subalgebra in $L^\infty(\mathbb{T})$.

(2) I think that $(\varphi_{\varepsilon+1}) \in L^\infty(\mathbb{T})$, since H^2 to be fixed $M_{(\varphi_{\varepsilon+1})}$ when only if $(\varphi_{\varepsilon+1}) \in H^\infty$.

Proof: (1) It can show this when $(\varphi_{\varepsilon+1}), (\psi_{\varepsilon+1}) \in H^\infty$, hence $(\varphi_{\varepsilon+1})(\psi_{\varepsilon+1}) \in H^\infty$ which indicates that the other processes are clear. When $n \in \mathbb{Z}$, hence $(\varphi_{\varepsilon+1}\widehat{)(\psi_{\varepsilon+1}})(n) = \sum_{m=-\infty}^{\infty} (\widehat{\varphi_{\varepsilon+1}})(m)\,(\widehat{\psi_{\varepsilon+1}})(n-m)$. Where $n < 0$, hence $(\varphi_{\varepsilon+1}\widehat{)(\psi_{\varepsilon+1}})(n) = 0$, whenever $(\varphi_{\varepsilon+1})(\psi_{\varepsilon+1}) \in H^\infty$. (2) comes from (1).

Notation 1.7. Put $v := M_{\varepsilon_1}$and $u := v|_{H^2}$.

Note that v is a unitary element in C^*-Algebra $B(L^2(\mathbb{T}))$ and that for every $n \in \mathbb{Z}$ $\quad v(\varepsilon_n) = \varepsilon_{n+1}$. So v is the binary transformation based on $(\varepsilon_n)_{n\in\mathbb{Z}}$ for $L^2(\mathbb{T})$. Similarly, u is unilateral Transformation based on $((\varepsilon_n)_{n\in\mathbb{Z}}$ from H^2.

Theorem 1.8. Let $w \in B\left(L^2(\mathbb{T})\right)$, hence $wv = vw$ when $w = M_{(\varphi_{\varepsilon+1})}$ to any $(\varphi_{\varepsilon+1}) \in L^\infty(\mathbb{T})$.

Proof: We explain the first part only, while the rest is obvious. Put $wv = vw$. Take $(\psi_{\varepsilon+1}) \in \Gamma$, hence $M_{(\psi_{\varepsilon+1})}$ linear field of effects v^n $(n \in \mathbb{Z})$ then $M_{(\psi_{\varepsilon+1})}$ qualify to w. Let $(\psi_{\varepsilon+1})$ represents an arbitrary item in $L^\infty(\mathbb{T})$, which means it is serial $(\psi_{\varepsilon+1})_{n(n\in\mathbb{Z})} \subset \Gamma$ as in

$$\lim_{n\to\infty} \|(\psi_{\varepsilon+1}) - (\psi_{\varepsilon+1})_n\|_{L^2} = 0.$$

Then $\lim_{n\to\infty} \|w(\psi_{\varepsilon+1}) - w(\psi_{\varepsilon+1})_n\|_{L^2} = 0$ when w ongoing at $L^2(\mathbb{T})$. The partial sequence can be referred to when it is important the $(\psi_{\varepsilon+1})_{n(n\in\mathbb{Z})}$ is converging at $w(\psi_{\varepsilon+1})$. Put $(\varphi_{\varepsilon+1}) = w(\varepsilon_0)$. When $n \in \mathbb{Z}$,

$$w(\psi_{\varepsilon+1})_n = wM_{(\psi_{\varepsilon+1})_n}(\varepsilon_0) = M_{(\psi_{\varepsilon+1})_n}w(\varepsilon_0) = (\psi_{\varepsilon+1})_n(\varphi_{\varepsilon+1}) \qquad (4)$$

Then $w(\psi_{\varepsilon+1}) = (\psi_{\varepsilon+1})(\varphi_{\varepsilon+1}) = (\varphi_{\varepsilon+1})(\psi_{\varepsilon+1})$.

Put $E_n = \{\lambda \in \mathbb{T} | (\varphi_{\varepsilon+1})(\lambda)| > \|w\| + 1/n\}$. It is clear that E_n standard set, then each of them in $n \in \mathbb{N}$ can be

$$\|w\|^2\|\chi_{E_n}\|_{L^2}^2 \geq \|w(\chi_{E_n})\|_{L^2}^2 = \int |(\varphi_{\varepsilon+1})(\lambda)|^2 \chi_{E_n}(\lambda)d\lambda$$
$$\geq (\|w\| + 1/n)^2 \int \chi_{E_n}(\lambda)d\lambda = (\|w\| + 1/n)^2 \|\chi E_n\|_{L^2}^2 \qquad (5)$$

we can conclude that E_n its zero measurement. Then $\bigcup_{n=1}^{\infty} E_n = \{\lambda \in \mathbb{T} \big| |(\varphi_{\varepsilon+1})(\lambda)| > \|w\|\}$ its zero measurement, then $|(\varphi_{\varepsilon+1})(\lambda)| \leq \|w\|$. So $(\varphi_{\varepsilon+1}) \in L^\infty(\mathbb{T})$. In order to $w|_{L^\infty(\mathbb{T})} = M_{(\varphi_{\varepsilon+1})}$, one of the conclusions is $w = M_{(\varphi_{\varepsilon+1})}$ in $L^2(\mathbb{T})$ then $L^\infty(\mathbb{T})$ its $L^2 - norm$ it's a dense at $L^2(\mathbb{T})$.

Definition 1.9. (1) Put E the Borel group $\mathbb{T}$, which is called the domain K_E from projection M_{χ_E} to $L^2(\mathbb{T})$ Wiener spaces vectors to $L^2(\mathbb{T})$.

(2) Put $(\varphi_{\varepsilon+1})$ it is a standard component in $C^* - algebra\ L^\infty(\mathbb{T})$, to the finite directional subspace $(\varphi_{\varepsilon+1})H^2$ to $L^2(\mathbb{T})$ it's the vector space of Beurling.

We can notice $v(K_E) = K_E$. Leads to $M_{\chi_E}v = vM_{\chi_E}$ from 1.7 and when $(\varphi_{\varepsilon+1}) \in L^\infty(\mathbb{T})$ it's a unit component hence $(\varphi_{\varepsilon+1})H^2$ its fixed in v, however $v(\varphi_{\varepsilon+1})H^2 \neq (\varphi_{\varepsilon+1})H^2$. Let $v(\varphi_{\varepsilon+1})H^2 = (\varphi_{\varepsilon+1})H^2$, hence $v(H^2) = u(H^2)$, then that one-sided shift is my guess, it's a contradictions.

Theorem 1.10. Let K a limited subspace at $L^2(\mathbb{T})$ its fixed at v, hence:

(1) $v(K) = K \Leftrightarrow K$, represent Wiener space,

(2) $v(K) \neq K \Leftrightarrow K$, represent Beurling space,

Proof: Through our observation, as shown in the proof of the two results.

(1) I think $v(K) = K$, can put $(\alpha + 1)$ is the vertical projection in $L^2(\mathbb{T})$ to K. Hence K decrease v, we get $(\alpha + 1)v = v(\alpha + 1)$, it's from theorem 1.8 its existing $(\varphi_{\varepsilon+1}) \in L^\infty(\mathbb{T})$, at like $(\alpha + 1) = M_{(\varphi_{\varepsilon+1})}$, hence $(\alpha + 1)$ its dropping, then $(\varphi_{\varepsilon+1})$ can be $(\varphi_{\varepsilon+1}) = \chi_E$ when E it's a measurable group. Which means that K it's a Wiener spaces.

(2) Suppose that $v(K) \neq K$ represent the unit vector $(\varphi_{\varepsilon+1}) \in K$ such as $(\varphi_{\varepsilon+1}) \perp v(K)$. Hence $v^n(K) \in v(K)$ to any $n > 0$ can be

$$0 = \langle v^n((\varphi_{\varepsilon+1})), (\varphi_{\varepsilon+1})\rangle = \int \varepsilon_n(\lambda)|(\varphi_{\varepsilon+1})(\lambda)|^2 d\lambda.$$

So that $n \in \mathbb{Z} \setminus \{0\}$, it can $\int \varepsilon_n(\lambda)|(\varphi_{\varepsilon+1})(\lambda)|^2 d\lambda = 0$, & $|(\varphi_{\varepsilon+1})|^2 = \alpha$, to any $\alpha \in \mathbb{R}$, from Lemma1.5. Hence $L^2 - norm$ at $(\varphi_{\varepsilon+1})$ to be 1, that $\alpha = 1$, so $(\varphi_{\varepsilon+1})$ it is a standard component as $L^\infty(\mathbb{T})$. And in the simplest way $(\varphi_{\varepsilon+1})H^2 \subset K$. As well $(\varepsilon_n(\varphi_{\varepsilon+1}))_{n\in\mathbb{Z}}$ be orthogonal basis K, then $K = (\varphi_{\varepsilon+1})H^2$. And so K to be Beurling spaces.

Theorem 1.11. The branched areas of vectors are limited only to H^2 diversion from one side for movement u its insignificant spaces that do not change 0 & H^2.

Proof: I think that K it finite and extremely non-trivial subspace at H^2 diversion to u. Hence $\bigcap_{n=1}^{\infty} u^n(K) \subseteq \bigcap_{n=0}^{\infty} u^n(H^2) = 0$ & $K \neq 0$, thus $v(K) = u(K) \neq K$, then K represent the Berling space from Theorem 1.10. Like wise $H^2 \ominus K$ to be Beurling spaces. Since is to be unitary elements $(\varphi_{\varepsilon+1}), (\psi_{\varepsilon+1}) \in L^\infty(\mathbb{T})$ so,

$K = (\varphi_{\varepsilon+1})H^2$ & $H^2 \ominus K = (\psi_{\varepsilon+1})H^2$.

For every $n \geq 0$, to us $\varepsilon_n(\varphi_{\varepsilon+1}) \in K$ & $\varepsilon_n(\psi_{\varepsilon+1}) \in H^2 \ominus K$, then

$$\langle \varepsilon_n(\varphi_{\varepsilon+1}), (\psi_{\varepsilon+1})\rangle = \langle (\varphi_{\varepsilon+1}), \varepsilon_n(\psi_{\varepsilon+1})\rangle = 0$$

Then, $(\varphi_{\varepsilon+1})\overline{(\psi_{\varepsilon+1})}$ its zero Fourier transform, hence $(\varphi_{\varepsilon+1})\overline{(\psi_{\varepsilon+1})} = 0$. This can be discrepancy $(\varphi_{\varepsilon+1})$ & $(\psi_{\varepsilon+1})$ that is elements of unity. Hence unique boundaries of space vector in finite branches can be u is 0 & H^2.

Theorem 1.12. Put $f \in H^2$ do not fade hence $d\lambda(f^{-1}(\{0\})) = 0$.

Proof: Put K until $L^2 - norm$ partially locked vector space at H^2 it contains whole the elements $g \in H^2$ as well as $g\chi_{f^{-1}}(\{0\}) = 0$. Hence K its fixed to u so v & $\bigcap_{n=1}^{\infty} u^n(K) \subseteq \bigcap_{n=0}^{\infty} u^n(H^2) = 0$. Wherefore that $v(K) = K$, so there for $K = 0$, from that time $f \in K, f = 0$. Be on the wane .So $v(K) \neq K$, from Theorem1.10, $K = (\varphi_{\varepsilon+1})H^2$ to any standard component $(\varphi_{\varepsilon+1}) \in L^\infty(\mathbb{T})$. And so that $(\varphi_{\varepsilon+1})\chi_{f^{-1}}(\{0\}) = 0$. Where $\chi_{f^{-1}}(\{0\}) = 0$. Therefore $d\lambda\big(f^{-1}(\{0\})\big) = 0$.

2- Relationship between Toeplitz and Matrix Effects, Put $(\varphi_{\varepsilon+1})$ dropping at $L^2(\mathbb{T})$ to H^2.

Definition 2.1 Let $(\varphi_{\varepsilon+1}) \in L^\infty(\mathbb{T})$, so the Toeplitz effects $T_{(\varphi_{\varepsilon+1})}$ the tags $(\varphi_{\varepsilon+1})$ effects in,

$$T_{(\varphi_{\varepsilon+1})}: H^2 \to H^2\ ,\ f \longmapsto (\alpha+1)\{(\varphi_{\varepsilon+1})f\} \tag{6}$$

this pressure from $M_{(\varphi_{\varepsilon+1})}$ into H^2.

Obviously $\|T_{(\varphi_{\varepsilon+1})}\| \leq \|(\varphi_{\varepsilon+1})\|_\infty$ can be,

$$T_*: L^\infty(\mathbb{T}) \to B(H^2), (\varphi_{\varepsilon+1}) \longmapsto T_{(\varphi_{\varepsilon+1})} \tag{7}$$

Linear convergent function to $T^*_{(\varphi_{\varepsilon+1})} = T_{\overline{(\varphi_{\varepsilon+1})}}$. So that $\overline{(\varphi_{\varepsilon+1})} = (\varphi_{\varepsilon+1})$, after that $T_{(\varphi_{\varepsilon+1})}$ it's a self-help.

The Toeplitz matrix's $\left(a_{ij}\right)_{i,j\in\mathbb{N}}$ can be represents the matrix of fixed diagonals $\left(a_{ij}\right) = \left(a_{i+1,j+1}\right)$ for every i,j. The metric matrix can be written as:

$$\begin{matrix} a_0 & a_{-1} & a_{-2} & a_{-3} & \cdots \\ a_1 & a_0 & a_{-1} & a_{-2} & \cdots \\ a_2 & a_1 & a_0 & a_{-1} & \cdots \\ a_3 & a_2 & a_1 & a_0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{matrix}$$

It is clear from this that the topological matrix is determined in two ways $(a_n)_{n\in\mathbb{Z}} \subset \mathbb{C}$. Where $a_{ij} = a_{i-j}$. As the Toeplitz trigger moves with a one-sided shift u, it is clear that the matrix of a Toeplitz effects $T_{(\varphi_{\varepsilon+1})}$ where about the base $(\varepsilon_n)_{n\in\mathbb{N}}$ it's a Topological matrix. Furthermore, and the two-sided series corresponding to the matrix, $T_{(\varphi_{\varepsilon+1})}$ to be $\left((\widehat{\varphi_{\varepsilon+1}})(n)\right)_{n\in\mathbb{N}}$.

Remark 2.2. We can prove it if $a \in (H^2)$ it is a matrix for that base of $(\varepsilon_n)_{n\in\mathbb{N}}$ represents the Toeplitz matrix, where a it can a Toeplitz effects.

The problems facing this theory result from the fact that the linear map T_* not double. Especially, $T_{(\varphi_{\varepsilon+1})}T_{(\psi_{\varepsilon+1})}$ seldom equal to $T_{(\varphi_{\varepsilon+1})(\psi_{\varepsilon+1})}$. For instance, if $(\varphi_{\varepsilon+1}) = \varepsilon_1$, $(\psi_{\varepsilon+1}) = \varepsilon_{-1}$, then $T_{(\varphi_{\varepsilon+1})} = u$, $T_{(\psi_{\varepsilon+1})} = u^*$. Where $u^*(\varepsilon_0) = 0$, $uu^* \neq 1$, that $T_{(\varphi_{\varepsilon+1})}T_{(\psi_{\varepsilon+1})} = T_{\varepsilon_0} = 1$, then $T_{(\varphi_{\varepsilon+1})}T_{(\psi_{\varepsilon+1})} \neq T_{(\varphi_{\varepsilon+1})(\psi_{\varepsilon+1})}$.

However, there is an explicit formula for a binary product matrix.

Put $\left(a_{i-j}\right)_{i,j\in\mathbb{N}}$, $\left(b_{i-j}\right)_{i,j\in\mathbb{N}}$ which represent the matrices of the Toeplitz effect $T_{(\varphi_{\varepsilon+1})}$, $T_{(\psi_{\varepsilon+1})}$ respectively. When $\left(c_{i,j}\right)_{i,j\in\mathbb{N}}$ represents the matrix of operating effects $T_{(\varphi_{\varepsilon+1})}T_{(\psi_{\varepsilon+1})}$, then everyone i,j,

$$c_{i+1,j+1} = c_{i,j} + a_{i+1}b_{-j-1} \tag{8}$$

The above equation can be called the multiplication in this article [3]. This indicates direct evidence.

This equation gives us an important property of $T_{(\varphi_{\varepsilon+1})}T_{(\psi_{\varepsilon+1})}$ its Toeplitz effect.

Theorem 2.3. Put $(\varphi_{\varepsilon+1}), (\psi_{\varepsilon+1}) \in L^\infty(\mathbb{T})$. That $(\psi_{\varepsilon+1}) \in H^\infty$, so $T_{(\varphi_{\varepsilon+1})(\psi_{\varepsilon+1})} = T_{(\varphi_{\varepsilon+1})}T_{(\psi_{\varepsilon+1})}$, so $T_{\overline{(\psi_{\varepsilon+1})}(\varphi_{\varepsilon+1})} = T_{\overline{(\psi_{\varepsilon+1})}}T_{(\varphi_{\varepsilon+1})}$. Conversely, the $T_{(\varphi_{\varepsilon+1})}T_{(\psi_{\varepsilon+1})}$ can be Toeplitz effect, that $\overline{(\varphi_{\varepsilon+1})}\ or\ (\psi_{\varepsilon+1}) \in H^\infty$, so $T_{(\varphi_{\varepsilon+1})}T_{(\psi_{\varepsilon+1})} = T_{(\varphi_{\varepsilon+1})(\psi_{\varepsilon+1})}$.

Proof: Let $(\psi_{\varepsilon+1}) \in H^\infty$, clear $(\psi_{\varepsilon+1})H^2 \subseteq H^2$, so that $f \in H^2$ thereafter

$$T_{(\varphi_{\varepsilon+1})}T_{(\psi_{\varepsilon+1})}(f) = (\alpha+1)\big(\varphi_{\varepsilon+1}(p\psi_{\varepsilon+1}f)\big) = (\alpha+1)(\varphi_{\varepsilon+1}\psi_{\varepsilon+1}f) = T_{(\varphi_{\varepsilon+1})(\psi_{\varepsilon+1})}$$

thereafter $T_{(\varphi_{\varepsilon+1})}T_{(\psi_{\varepsilon+1})} = T_{(\varphi_{\varepsilon+1})(\psi_{\varepsilon+1})}$. Then $T_{(\overline{\varphi_{\varepsilon+1}})}T_{(\psi_{\varepsilon+1})} = T_{(\overline{\varphi_{\varepsilon+1}})(\psi_{\varepsilon+1})}$, thereafter,

$$T_{(\overline{\psi_{\varepsilon+1}})}T_{(\varphi_{\varepsilon+1})} = T^*_{(\psi_{\varepsilon+1})}T^*_{(\overline{\varphi_{\varepsilon+1}})} = T^*_{(\overline{\varphi_{\varepsilon+1}})(\psi_{\varepsilon+1})} = \left(T_{(\overline{\varphi_{\varepsilon+1}})}T_{(\psi_{\varepsilon+1})}\right)^* = T_{(\overline{\psi_{\varepsilon+1}})(\varphi_{\varepsilon+1})}$$

If we assume the opposite to $T_{(\varphi_{\varepsilon+1})(\psi_{\varepsilon+1})}$ its aToeplitz effect. Thus, we find that its matrix represents the Toeplitz matrix, which is the multiplication equation, $a_{i+1}b_{-j-1} = 0$ for every i, j. Leads to $a_{i+1} = 0$ for every $i \geq 0$ or $b_{-j-1} = 0$, for every $j \geq 0$ which gives the desired result.

We find that the following examples show us the Theorem 2.3. We put (a, b, c, d), and $e \in \mathbb{C}$ where $(\psi_{\varepsilon+1}) = a\varepsilon_0 + b\varepsilon_1$, $(\varphi_{\varepsilon+1}) = c\varepsilon_{-1} + d\varepsilon_0 + e\varepsilon_1$. So $(\psi_{\varepsilon+1}) \in \Gamma_+ \subseteq H^\infty$, $(\varphi_{\varepsilon+1}) \in L^\infty(\mathbb{T})$. Then a matrix $(\psi_{\varepsilon+1})$ of $T_{(\psi_{\varepsilon+1})}$ can be:

$$(\psi_{\varepsilon+1}) = \begin{matrix} a & 0 & 0 & 0 & \dots \\ b & a & 0 & 0 & \dots \\ 0 & b & a & 0 & \dots \\ 0 & 0 & b & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{matrix}$$

Similarly, the matrix Φ of $T_{(\varphi_{\varepsilon+1})}$ can be:

$$\Phi = \begin{matrix} d & c & 0 & 0 & \dots \\ e & d & c & 0 & \dots \\ 0 & e & d & c & \dots \\ 0 & 0 & e & d & \dots \\ \dots & \dots & \dots & \dots & \dots \end{matrix}$$

It is clear from this that the metric Toeplitz is product[3]

$$\Phi(\psi_{\varepsilon+1}) = \begin{matrix} ab+bc & ac & 0 & 0 & \dots \\ ae+bd & ad+bc & ac & 0 & \dots \\ be & ae+bd & ad+bc & ac & \dots \\ 0 & be & ae+bd & ad+bc & \dots \\ \dots & \dots & \dots & \dots & \dots \end{matrix}$$

that it's a Toeplitz matrix. When $(\varphi_{\varepsilon+1})(\psi_{\varepsilon+1}) = ac\varepsilon_{-1} + (ad+bc)\varepsilon_0 + (ae+bd)\varepsilon_1 + be\varepsilon_2$ which means that it is the matrix of Toeplitz influences $T_{(\varphi_{\varepsilon+1})(\psi_{\varepsilon+1})}$.

Corollary 2.4. There are no zero denominators for all the Toeplitz effects. Mostly, whether $(\varphi_{\varepsilon+1})(\psi_{\varepsilon+1}) \in L^\infty(\mathbb{T})$ [7], so

$$T_{(\varphi_{\varepsilon+1})}T_{(\psi_{\varepsilon+1})} = 0 \Leftrightarrow T_{(\varphi_{\varepsilon+1})} = 0 \; or \; T_{(\psi_{\varepsilon+1})} = 0$$

Proof: Implied means $\Leftarrow$ its petty, therefore, the opposite can be proven. When 0 its Toeplitz effects, can be $(\overline{\varphi_{\varepsilon+1}})$ or $(\psi_{\varepsilon+1}) \in H^\infty(\subset H^2)$, $(\varphi_{\varepsilon+1})(\psi_{\varepsilon+1}) = 0$, from Theorem 2.3. We can see that $(\overline{\varphi_{\varepsilon+1}}) \in H^2$, where $(\psi_{\varepsilon+1}) = 0$, when $(\psi_{\varepsilon+1}) \in H^2$, so $(\varphi_{\varepsilon+1}) = 0$. Consequently $T_{(\varphi_{\varepsilon+1})} = 0$, or $T_{(\psi_{\varepsilon+1})} = 0$.

This conclusion leads us to the following question:

Question 2.5. Put $(\varphi_{\varepsilon+1})_i \in L^\infty(\mathbb{T}), (i = 1,2, \dots, n)$, so that

$$T_{(\varphi_{\varepsilon+1})_1}T_{(\varphi_{\varepsilon+1})_2} \dots T_{(\varphi_{\varepsilon+1})_n} = 0$$

So it is important to have an index of this type i, when $T_{(\varphi_{\varepsilon+1})_i} = 0$?

Therefore, case means an explanation of the Corollary 2.4, which was solved by Alexandru Aleman and Dragan Vukoti´c [4]. This is from clarification that the previous problem includes the positive result for each n.

Theorem 2.6. Let $(\varphi_{\varepsilon+1}) \in H^\infty$, where $(\varphi_{\varepsilon+1})$ not scalar, so $T_{(\varphi_{\varepsilon+1})}$ it has no intrinsic value.

Proof: Assuming that $f \in H^2, \lambda \in \mathbb{C}$, $\left(T_{(\varphi_{\varepsilon+1})} - \lambda\right)(f) = 0$

So that $\big((\varphi_{\varepsilon+1})-\lambda\big)(f)=0$. Since $(\varphi_{\varepsilon+1})-\lambda\in H^2$, since it does not represent the zero element, that a set $\big\{\zeta\in\mathbb{C}\big|\big((\varphi_{\varepsilon+1})-\lambda\big)(\zeta)=0\big\}$ is a measurement 0. Thus $f=0$.

If we put A is a unit $C^*-algebra$ with the unit 1, when $a\in A$, so that spectrum $\sigma(a)$ of a at A can define by

$$\sigma(a)=\{\lambda\in\mathbb{C}|\lambda 1-a \text{ There is no reflection } at\ A\}$$

So that spectral radius of a is

$$r(a)=\sup_{\lambda\in\sigma(a)}|\lambda|.$$

Theorem 2.7. Put $(\varphi_{\varepsilon+1})\in L^\infty(\mathbb{T})$, $\sigma(\varphi_{\varepsilon+1})$ which refers to the spectrum to $(\varphi_{\varepsilon+1})$ at $L^\infty(\mathbb{T})$. So

$\sigma(\varphi_{\varepsilon+1})\subseteq\sigma\big(T_{(\varphi_{\varepsilon+1})}\big),\ r\big(T_{(\varphi_{\varepsilon+1})}\big)=\big\|T_{(\varphi_{\varepsilon+1})}\big\|=\|(\varphi_{\varepsilon+1})\|_\infty.$

Proof: To display it $\sigma(\varphi_{\varepsilon+1})\subseteq\sigma\big(T_{(\varphi_{\varepsilon+1})}\big)$, we just show that $T_{(\varphi_{\varepsilon+1})}$ be invertible at $B(H^2)$. So $(\varphi_{\varepsilon+1})$ its invertible at $L^\infty(\mathbb{T})$. Sure that deficiency results from $T_{(\varphi_{\varepsilon+1})}-\lambda=T_{(\varphi_{\varepsilon+1-\lambda})}$ where $\lambda\in\mathbb{C}$. Then can put $T_{(\varphi_{\varepsilon+1})}$ it not inverse, so that $m\coloneqq\big\|T^{-1}_{(\varphi_{\varepsilon+1})}\big\|$. To any $f\in H^2$,

$\|f\|=\big\|T^{-1}_{(\varphi_{\varepsilon+1})}T_{(\varphi_{\varepsilon+1})}(f)\big\|\leq m\big\|T_{(\varphi_{\varepsilon+1})}(f)\big\|.$

From this we can conclude $n\in\mathbb{Z}$.

$$\big\|M_{(\varphi_{\varepsilon+1})}(\mathcal{E}_n f)\big\|=\|(\varphi_{\varepsilon+1})\mathcal{E}_n f\|=\|(\varphi_{\varepsilon+1})\|\geq\big\|T_{(\varphi_{\varepsilon+1})}(f)\big\|\geq\frac{\|f\|}{m}=\frac{\|\mathcal{E}_n f\|}{m}$$

Since a functions $\mathcal{E}_n f$ is L^2-norm dense at $L^2(\mathbb{T})$. Then Γ is L^2-norm dense at $L^2(\mathbb{T})$. Then so that $g\in L^2(\mathbb{T})$ that $\big\|M_{(\varphi_{\varepsilon+1})}(g)\big\|\geq\|g\|/m$, and $M^*_{(\varphi_{\varepsilon+1})}M_{(\varphi_{\varepsilon+1})}\geq m^{-2}>0$. Then $M^*_{(\varphi_{\varepsilon+1})}M_{(\varphi_{\varepsilon+1})}$ is not inverse by $M_{(\varphi_{\varepsilon+1})}$, $M_{(\varphi_{\varepsilon+1})}$ not invers . So that $M_*:L^\infty(\mathbb{T})\to B\big(L^2(\mathbb{T})\big)$ is symmetric $*$-homomorphism. $(\varphi_{\varepsilon+1})$ it has no inverse at $L^\infty(\mathbb{T})$.

We can put $(\varphi_{\varepsilon+1})$ hypothetical at $L^\infty(\mathbb{T})$. So that $\sigma(\varphi_{\varepsilon+1})\subseteq\sigma\big(T_{(\varphi_{\varepsilon+1})}\big)$, then

$\big\|T_{(\varphi_{\varepsilon+1})}\big\|\leq\|(\varphi_{\varepsilon+1})\|=r(\varphi_{\varepsilon+1})\leq r\big(T_{(\varphi_{\varepsilon+1})}\big)\leq\big\|T_{(\varphi_{\varepsilon+1})}\big\|$, then $\big\|T_{(\varphi_{\varepsilon+1})}\big\|=r\big(T_{(\varphi_{\varepsilon+1})}\big)=\|(\varphi_{\varepsilon+1})\|$.

The Notation 2.8. Let H is a Hilbert space, $x,y\in H$ from that can be determined $x\otimes y^c$ at H by $x\otimes y^c(z)=\langle z,y\rangle_H x$, $z\in H$. So that $x\otimes y^c$, is a von Neumann-schatten products of x,y.

Where it can be seen easily $x\otimes y^c$ it represents a limited first-order effect when x,y non-trivial. Other essential properties of von Neumann-Schatten products are found in [2].

The Notation 2.9. Let H is a Hilbert space, can refer to $K(H)$, $F(H)$ elements for all built-in effects and the set of all finite rank effects at H respectively.

Theorem 2.10. Put $(\varphi_{\varepsilon+1})\in L^\infty(\mathbb{T})$, so $T_{(\varphi_{\varepsilon+1})}$ compressed if and only if $(\varphi_{\varepsilon+1})=0$.

Proof: Note that first

$$u^{*n}(\varepsilon_m)=\begin{cases}\varepsilon_{m-n} & if\ m\geq n\\ 0 & if\ m<n\end{cases}$$

Then, put $f\in H^2$, so

$$\|u^{*n}(f)\|^2=\|\textstyle\sum_{m=n}^{\infty}\langle f,\varepsilon_m\rangle\,\varepsilon_{m-n}\|^2=\sum_{m=n}^{\infty}|\langle f,\varepsilon_m\rangle|^2 \qquad (9)$$

Then, $u^{*n}(f)\to 0,(n\to\infty)$ at H^2. When $v\in F(H^2)$, can find $f_1,\ldots,f_n$ and $g_1,\ldots,g_n$ at H^2 which means $v=\sum_{j=1}^{n}f_j\otimes g_j^c$, hence, we find that any positive integer m, become $u^{*m}v=\sum_{j=1}^{n}u^{*m}\big(f_j\big)\otimes g_j^c$. Then, $\lim_{m\to\infty}\|u^{*m}v\|=0$

then $F(H^2)$ dense at $K(H^2)$, $\lim_{m\to\infty}\|u^{*m}w\| = 0$, to any $w \in K(H^2)$. Note that $u^*T_{(\varphi_{\varepsilon+1})}u = T_{\varepsilon-1}T_{(\varphi_{\varepsilon+1})}T_{\varepsilon 1} = T_{\varepsilon-1(\varphi_{\varepsilon+1})\varepsilon 1} = T_{(\varphi_{\varepsilon+1})}$ from Theorem 2.3, then if $T_{(\varphi_{\varepsilon+1})}$it is built in

$$\|T_{(\varphi_{\varepsilon+1})}\| = \|u^{*m}T_{(\varphi_{\varepsilon+1})}u^{*m}\| \le \|u^{*m}T_{(\varphi_{\varepsilon+1})}\| \; and \lim_{m\to\infty}\|u^{*m}T_{(\varphi_{\varepsilon+1})}\| = 0 \quad (10)$$

then $T_{(\varphi_{\varepsilon+1})} = 0$. From Theorem 2.7. can find that $(\varphi_{\varepsilon+1}) = 0$.

Lemma 2.11. Put $(\varphi_{\varepsilon+1}) \in C(\mathbb{T})$ and $(\psi_{\varepsilon+1}) \in L^\infty(\mathbb{T})$, so that $T_{(\varphi_{\varepsilon+1})}T_{(\psi_{\varepsilon+1})} - T_{(\varphi_{\varepsilon+1})(\psi_{\varepsilon+1})}$ and $T_{(\psi_{\varepsilon+1})}T_{(\varphi_{\varepsilon+1})} - T_{(\psi_{\varepsilon+1})(\varphi_{\varepsilon+1})}$ are compact effects.

Proof: When $K(H^2)$ is self-adjoint, we only have to show that $T_{(\psi_{\varepsilon+1})}T_{(\varphi_{\varepsilon+1})} - T_{(\psi_{\varepsilon+1})(\varphi_{\varepsilon+1})} \in K(H^2)$. Then Γ is dense in $C(\mathbb{T})$ and the map T_* is a linear, we may suppose that $(\varphi_{\varepsilon+1}) = \varepsilon_n$ for some $n \in \mathbb{Z}$. Put $n \ge 0$, from Theorem 2.3. $T_{(\psi_{\varepsilon+1})}T_{\varepsilon_n} - T_{(\psi_{\varepsilon+1})\varepsilon_n} = 0 \in K(H^2)$.

Then became $T_{(\psi_{\varepsilon+1})}T_{\varepsilon_{-k}} - T_{(\psi_{\varepsilon+1})\varepsilon_{-k}}$ is compact for all $K \ge 1$, by induction at K. We put $f \in H^2$, so

$T_{(\psi_{\varepsilon+1})}T_{\varepsilon_{-1}}(f) = (\alpha+1)\big((\psi_{\varepsilon+1})(\alpha+1)(\varepsilon_{-1}f)\big) = (\alpha+1)\big((\psi_{\varepsilon+1})(\varepsilon_{-1}f - \langle f,\varepsilon_0\rangle\varepsilon_{-1})\big) = T_{(\psi_{\varepsilon+1})\varepsilon_{-1}}T_{\varepsilon_{-1}}f - \langle f,\varepsilon_0\rangle(\alpha+1)(\psi_{\varepsilon+1})\varepsilon_{-1}$. Then

$T_{(\psi_{\varepsilon+1})}T_{\varepsilon_{-1}} - T_{(\psi_{\varepsilon+1})\varepsilon_{-1}}$is an operator of rank not greater than one.

Suppose we show that in $T_{(\psi_{\varepsilon+1})}T_{\varepsilon_{-k}} - T_{(\psi_{\varepsilon+1})\varepsilon_{-k}}$ is compact for $(\psi_{\varepsilon+1}) \in L^\infty(\mathbb{T})$, for any $k \ge 1$. So

$$T_{(\psi_{\varepsilon+1})}T_{\varepsilon_{-k-1}} - T_{(\psi_{\varepsilon+1})\varepsilon_{-k-1}}$$

$$= \big(T_{(\psi_{\varepsilon+1})}T_{\varepsilon_{-k}} - T_{(\psi_{\varepsilon+1})\varepsilon_{-k}}\big)T_{\varepsilon_{-1}} + T_{(\psi_{\varepsilon+1})(\varepsilon_{-k})}T_{\varepsilon_{-1}} - T_{((\psi_{\varepsilon+1})(\varepsilon_{-k}))\varepsilon_{-1}} \qquad (11)$$

is compact.

3. The Toeplitz Algebra. Put $\mathbb{A}$ can be C^*-algebra denoted by any Toeplitz effects $T_{(\varphi_{\varepsilon+1})}$ with symbol $(\varphi_{\varepsilon+1}) \in C(\mathbb{T})$, where $\mathbb{A}$ is Toeplitz algebra.

Put $\mathbb{A}$ is a C^*-algebra, then its commutator ideal is the closed ideal generated by the commutators $[a,b] = ab - ba, (a,b) \in \mathbb{A}$. Clearly, the commutator ideal is the smallest closed ideal I in $\mathbb{A}$ such that $\mathbb{A}/I$ is abelian. First, we determine the commutator ideal of the Toeplitz algebra $\mathbb{A}$. [6].

Theorem 3.1. The commutator ideal of $\mathbb{A}$ is $K(H^2)$.

Proof: Observe first that if K is a closed vector subspace of H^2 invariant for $\mathbb{A}$, then K is reducing for u, and therefore by Theorem 1.11 $K = 0$ or H^2. Thus, $\mathbb{A}$ is an irreducible sub algebra of $B(H^2)$.

Now, set $(\alpha+1) = [u^*, u] = 1 - uu^*$. Obviously, $(\alpha+1)$ is of rank 1 and belongs to the commutator ideal I of $\mathbb{A}$. Since finite-rank operators are compact, $(\alpha+1) \in \mathbb{A} \cap K(H^2)$. [2] (and the irreducibility in one of its assumption), it follows that $K(H^2) \subseteq \mathbb{A}$. Since $\mathbb{A}/K(H^2)$ is generated by the elements

$$T_{(\varphi_{\varepsilon+1})} + K(H^2), \qquad (\varphi_{\varepsilon+1}) \in C(\mathbb{T})$$

which are commuting and normal by Lemma 2.10, $\mathbb{A}/K(H^2)$ is abelian. Because $K(H^2)$is simple, $1 = K(H^2)$.

Theorem 3.2. The map

$$\Psi: C(\mathbb{T}) \to \mathbb{A}/K(H^2), (\varphi_{\varepsilon+1}) \to T_{\varphi_{\varepsilon+1}} + K(H^2)$$

is $a*$isomorphism

Proof: Put $(\varphi_{\varepsilon+1}), (\psi_{\varepsilon+1}) \in C(\mathbb{T})$, from Lemma 2.10

$$\Psi\big((\varphi_{\varepsilon+1})(\psi_{\varepsilon+1})\big) = T_{(\varphi_{\varepsilon+1})(\psi_{\varepsilon+1})} + K(H^2) = T_{(\varphi_{\varepsilon+1})}T_{(\psi_{\varepsilon+1})} + K(H^2) = \Psi(\varphi_{\varepsilon+1})\Psi(\psi_{\varepsilon+1})$$

Then Ψ is multiplicative so that the map T_* it is linear and convergent so is Ψ. Consequently Ψ it's a$*$-homomorphism obviously the elements $T_{(\varphi_{\varepsilon+1})} + K(H^2), \big((\varphi_{\varepsilon+1}) \in C(\mathbb{T})\big)$ becomes $\mathbb{A}/K(H^2)$, so that Ψ is surjective. When $\Psi(\varphi_{\varepsilon+1}) = T_{(\varphi_{\varepsilon+1})} + K(H^2) = K(H^2)$, then $T_{(\varphi_{\varepsilon+1})} \in K(H^2)$, and from Theorem 2.9 $(\varphi_{\varepsilon+1}) = 0$. Ψ can be injective.

Now, we mention the indispensable items in the player theorist's tool-kit; that index and main Domain. Put X, Y be Banach distances. $w \in B(X,Y)$ is Fredholm if $dim\ (ker\ w)$ and $codim_{\mathrm{Y}}(\mathrm{w\ (X)})$ limited. We define the index of w can be

$ind\ (w) = dim(ker\ w) - codim_{\mathrm{Y}}(\mathrm{w\ (X)})$.

We can use the following from [2] to calculate the Friedholm indices.

Fact 3.3. Put X, Y, Z can be Banach spaces, and put $w_1 = X \to Y$ and $w_2 = Y \to Z$ be Fredholm operators. So $w_2 w_1$ is Fredholm and

$$ind(w_2 w_1) = ind(w_2) + ind(w_1)$$

Since all the factors between finite-dimensional spaces are Fredholm, we assume that X is an infinite-dimensional Banach space. We have a useful characterization of the Friedholm operators.

Fact 3.4. If $w \in B(X)$ then w can be Fredholm $\Leftrightarrow$ $w + K(X)$ is invertible at $B(X)/K(X)$. The basic spectrum $\sigma_e(w_1)$ of $w \in B(X)$ known as

$$\sigma_e(w_1) = \{\lambda \in \mathbb{C} | w - \lambda\ is\ not\ \text{Fredholm}\}$$

If $\pi: B(X) \to B(X)/K(X)$ is the quotient map, it is clear that $\sigma_e(w_1) = \sigma\big(\pi(w)\big)$ by Atkinson characterization.

Corollary 3.5 Put $(\varphi_{\varepsilon+1}) \in C(\mathbb{T})$, so $T_{(\varphi_{\varepsilon+1})}$ is Fredholm if and only if $(\varphi_{\varepsilon+1})$ it doesn't disappear anywhere.

Proof: From Atkinson characterization, Theorem 3.2,

$T_{(\varphi_{\varepsilon+1})}$is Fredholm $\Leftrightarrow$ $T_{(\varphi_{\varepsilon+1})} + K(H^2)\ is\ invertible\ at\ B(H^2)/K(H^2)$

$$\Leftrightarrow\ T_{(\varphi_{\varepsilon+1})} + K(H^2)\ is\ invertible\ at\ \mathbb{A}(H^2)/K(H^2)$$

Then from Theorem 3.2, $T_{(\varphi_{\varepsilon+1})}$ is Fredholm if and only if $(\varphi_{\varepsilon+1})$ is invertible at $C(\mathbb{T})$.

Corollary 3.6. Put $(\varphi_{\varepsilon+1}) \in C(\mathbb{T})$, so $\sigma_e\big(T_{(\varphi_{\varepsilon+1})}\big) = (\varphi_{\varepsilon+1})(\mathbb{T})$

Proof: By the Atkinson characterizaton and Theorem 3.2 $\sigma_e\big(T_{(\varphi_{\varepsilon+1})}\big) = \sigma\Big(\pi\big(T_{(\varphi_{\varepsilon+1})}\big)\Big) = \sigma(\varphi_{\varepsilon+1}) = (\varphi_{\varepsilon+1})(\mathbb{T})$.

Then, the continuous symbol Toeplitz effects connected the basic spectrum. Especially, $\sigma_e(u) = \sigma\big(T_{\varepsilon_1}\big) = \mathbb{T}$.

Then, we calculate the Fredholm index of Toeplitz effects $\big(T_{\varepsilon_n}\big)$. Since $\ker u = 0$ and $u(H^2)$ is the closed linear span of the set $\{\ \varepsilon_n |\ n \geq 1\}$, so $ind\big(T_{\varepsilon_1}\big) =\ ind(u) = 0 - 1 = -1$. So, the unilateral shift u has index -1. If $n \in \mathbb{N}$, then $ind\big(T_{\varepsilon_n}\big) = ind(u^n) = n\ ind(u) = -n$, where Fact 3.3 has been used in the second equality. By, the equality $T_{\varepsilon_{-n}} T_{\varepsilon_n} = T_1$ one also infers that $ind\big(T_{\varepsilon_{-n}}\big) = n = -(-n)$.

And when we put $(\varphi_{\varepsilon+1}) \in C(\mathbb{T})$ is a reversible element with $wn(\varphi_{\varepsilon+1}) = n$. Then $(\varphi_{\varepsilon+1}) = \varepsilon_n e^{(\psi_{\varepsilon+1})}$ to any $(\psi_{\varepsilon+1}) \in C(\mathbb{T})$ and $T_{(\varphi_{\varepsilon+1})} = T_{\varepsilon_n} T_{e^{(\psi_{\varepsilon+1})}}$ when $n \geq 0$, $T_{\emptyset} = T_{\epsilon_n} T_{e^{(\psi_{\varepsilon+1})}}$, when $n < 0$, from Theorem 2.3. Then

$$ind\big(T_{(\varphi_{\varepsilon+1})}\big) = ind(\varepsilon_n) + inde^{(\psi_{\varepsilon+1})} = -n$$

So $T_{e^{(\psi_{\varepsilon+1})}}$ it is from index zero with the proof of Lemma 3.8. Below.

Since the winding number of $(\varphi_{\varepsilon+1})$ is unchanged as long as the homotopy class is fixed, so is the "analytic invariant", the Fredholm index of a Toeplitz effects $T_{(\varphi_{\varepsilon+1})}$.

We will generalize this result, which is the simplest case of the Atiyah-Singer index theorem on odd-dimensional manifolds.

Theorem 3.7. If $(\varphi_{\varepsilon+1})$ represents a reversible element at $C(\mathbb{T})$ so that

$$ind(T_{(\varphi_{\varepsilon+1})}) = -wn(\varphi_{\varepsilon+1})$$

Where $wn(\varphi_{\varepsilon+1})$ is the winding number of $(\varphi_{\varepsilon+1})$ with respect to the origin.

The winding number of the continuous function $(\varphi_{\varepsilon+1})\colon \mathbb{T} \to \mathbb{C}\backslash\{0\}$ is the unique integer n given by next lemma.

Lemma 3.8. If $(\varphi_{\varepsilon+1})$ is an inverse function in $C(\mathbb{T})$, so that so that there exists a unique integer $n \in \mathbb{Z}$ such that $(\varphi_{\varepsilon+1}) = \varepsilon_n e^{(\psi_{\varepsilon+1})}$ for some $(\psi_{\varepsilon+1}) \in C(\mathbb{T})$.

Proof: Here we show the uniqueness of the integer n (for the existence of n and $(\psi_{\varepsilon+1})$; can find in [2]). We need only show that if $\varepsilon_n = e^{(\psi_{\varepsilon+1})}$ to any $(\psi_{\varepsilon+1}) \in C(\mathbb{T})$, so that $n = 0$. Then

$$\alpha\colon [0,1] \to \mathbb{Z}, t \mapsto ind\left(T_{e^{t(\psi_{\varepsilon+1})}}\right)$$

is continuous and has discrete range and connected domain, so it is necessarily constant. Hence

$$-n = ind(T_{e^{(\psi_{\varepsilon+1})}}) = \alpha(1) = \alpha(0) = ind(T_1) = ind(1) = 0$$

Corollary 3.9. Put $(\varphi_{\varepsilon+1}) \in C(\mathbb{T})$ is an invertible element. So the following conditions are equivalent.

1- $(T_{(\varphi_{\varepsilon+1})})$ is invertible.

2- $ind(T_{(\varphi_{\varepsilon+1})}) = 0$.

3- $(\varphi_{\varepsilon+1}) = e^{(\psi_{\varepsilon+1})}$ to any $(\psi_{\varepsilon+1}) \in C(\mathbb{T})$.

Proof: Can need only to show the implication (3) ⇒ (1) since the others are obvious from the preceding result.

The first, if $\rho = \sum_{|n|\le N} \lambda_n \varepsilon_n \in \Gamma$, writes $\rho^{/} = \sum_{n=0}^{N} \lambda_n \varepsilon_n$, and $\rho^{//} = \sum_{n=1}^{N} \lambda_{-n}\varepsilon_{-n}$, so $\rho = \rho^{/} + \rho^{//}$, and $\rho^{/}, \overline{\rho^{//}} \in H^\infty$. From (1) of Lemma 1.6 can be $e^{\rho^{/}}, e^{\overline{\rho^{//}}} \in H^\infty$. From Theorem 2.3 can be $T_{e^{\rho^{/}}}$ and $T_{\overline{e^{\rho^{//}}}}$ are invertible with inverses $T_{e^{-\rho^{/}}}$ and $T_{\overline{e^{-\rho^{//}}}}$ respectively.

Then if $(\psi_{\varepsilon+1}) \in C(\mathbb{T})$ is an arbitrary element, then by the density of Γ at $C(\mathbb{T})$, we can choose a trigonometric polynomial ρ as above such that

$\left\|1 - e^{(\psi_{\varepsilon+1})-\rho}\right\| < 1$. Then $T_{e^{(\psi_{\varepsilon+1})}} = T_{e^{\rho^{//}} e^{(\psi_{\varepsilon+1})-\rho} e^{\rho^{/}}} = T_{e^{\rho^{//}}} T_{e^{(\psi_{\varepsilon+1})-\rho}} T_{e^{\rho^{/}}}$. Since $\left\|1 - T_{e^{(\psi_{\varepsilon+1})-\rho}}\right\| = \left\|T_{1-e^{(\psi_{\varepsilon+1})-\rho}}\right\| = \left\|1 - e^{(\psi_{\varepsilon+1})-\rho}\right\| < 1$, the effect $T_{e^{(\psi_{\varepsilon+1})-\rho}}$ is invertible. Then $T_{e^{(\psi_{\varepsilon+1})}}$ is a product of invertible operators and is therefore invertible.

We have an application of the baby index theorem to the spectral theory of Toeplitz effects.

Theorem 3.10. The spectrum of a Toeplitz effects with continuous symbol is connected.

Proof: Put $(\varphi_{\varepsilon+1}) \in C(\mathbb{T})$, so by the baby index theorem and Corollary 3.8,

$$\sigma(T_{(\varphi_{\varepsilon+1})}) = (\varphi_{\varepsilon+1})(\mathbb{T}) \cup \{\lambda \in \mathbb{C} | wn((\varphi_{\varepsilon+1}) - \lambda) \neq 0\} \tag{12}$$

And therefore $\sigma(T_{(\varphi_{\varepsilon+1})})$ is a compact set consisting of a continuous compressed curve $(\varphi_{\varepsilon+1})(\mathbb{T})$ and some holes, so $\sigma(T_{(\varphi_{\varepsilon+1})})$ is connected.

In question 2.5, a natural question arises from this theory.

Question 3.11. Is the spectrum of the Toeplitz effects necessarily connected?

This question was asked by Halmos, then answered by Harold Widom [5] by proving that after theory.

Any Toeplitz effects is a continuum spectrum.

Remark 3.12. Since $C(\mathbb{T})$ is a C^*-algebra was created by ε_1, $\mathbb{A}$ resulted from unilateral transformation $u = T_{\varepsilon_1}$.

So that from (Wolde von Neumann). If W is isometric over Hilbert area H, then W is a Unitary, or direct sum of unilateral transformation copies, or direct sum of unitary and copies of unilateral transformation.

Remark 3.13. Put α to be a prime number. We refer by $u^{(\alpha)}$ the direct sum of α copies of unilateral transformation. Then says that the measurement W is of the form

$$W = u^{(\alpha)} \oplus U$$

Where U is a unit (may be empty).

Proof: We may assume that W is neither a unitary nor a sum of copies of a one-sided transformation. Puts

$$K = \bigcap_{n=0}^{\infty} W^n (H)$$

Then, $W(K) = K$, and thus K reduces W. We put $W = w \oplus \acute{w}$ where w and $\acute{w}$ is Compressions of W to K and $K^{\perp}$ respectively. Then the w is isometric, the equation $W(K) = K$ means that w is sectarian, so w is unitary. Therefore, we only need to prove that $\acute{w}$ is a direct amount From copying the shift by one u.

Now, set

$$L = (W(H))^{\perp}$$

So that, $W^n(L) \subseteq W(H) = L^{\perp}$, to any $n > 0$, then $m, n \in \mathbb{N}$ and $m \neq n$, then $W^m(L) \perp W^n(L)$. We claim it

$\oplus_{n=0}^{\infty} W^n(L) = K^{\perp}$

Clearly, $W^n(L) \perp W^n(L^{\perp}) = W^{n+1}(H)$. Since $K \subseteq W^{n+1}(H)$, one concludes that $W^n(L) \subseteq K^{\perp}$,and therefore $\oplus_{n=0}^{\infty} W^n(L) \subseteq K^{\perp}$. To display $\oplus_{n=0}^{\infty} W^n(L) \supseteq K^{\perp}$, it is enough to show that if $x \in H$, such that $x \perp W^n(L)$, to any $n \in \mathbb{N}$, then $x \in K$. This is clearly true for $n = 0$. If $x \in W^n(H)$, then there is $y \in H$, such that $x = W^n(y)$. Since $W^n(y) \perp W^n(L)$, then $y \subseteq L^{\perp} = W(H)$, and therefore $x \in W^{n+1}(H)$.

Let E be an orthogonal basis for L. Since W is an isometric, $\bigcup_{n=0}^{\infty} W^n(E)$ is an orthogonal basis of $K^{\perp}$. For each $e \in E$, make L_e be the Hilbert subspace of $K^{\perp}$ having $\left(W^n(e)\right)_{n=0}^{\infty}$ as orthogonal basis. Then

$$K^{\perp} = \bigoplus_{e \in E} L_e$$

each L_e is constant with respect to W, and the sympathy W_e from W to L_e is the one-sided transformation, and

$$\acute{w} = \bigoplus_{e \in E} W_e$$

Before moving on to Coburn's theorem, we recall the basic tool in operator theory [2].

Fact 3.14 (Functional Calculus). Put B be a unital C^*-algebra, a be a natural element in B and $z: \sigma(a) \to \mathbb{C}$ be the imputation map, [5]. Then there exists a unique unital isometric $*$-homomorphism

$$\Theta_a: C\big(\sigma(a)\big) \to B$$

such that $\Theta_a\,(z) = a$ and the $image$ $\Theta_a\left(C\big(\sigma(a)\big)\right)$ is the C^*-algebra generated by the unit of B and a.

The map Θ_a it is called functional calculus in a. Note that if $f \in C\big(\sigma(a)\big)$, then we write $\Theta_a(f) = f(a)$.

lemma 1.15. If w is aunitary in aunitary in a C^*-algebra B. Then there is aunique unit $*$-homomorphism $(\varphi_{\varepsilon+1}): C(\mathbb{T}) \to B$ such that $(\varphi_{\varepsilon+1})(z) = w$.

Proof: Note that $\sigma(w) \subseteq \mathbb{T}$ and that the map

$$i : C(\mathbb{T}) \to C\big(\sigma(w)\big), \qquad f \mapsto f|_{\sigma(w)}$$

is a unital $*$-homomorphism. One gets $(\varphi_{\varepsilon+1})$ by setting $(\varphi_{\varepsilon+1}) = \Theta_w\, oi$. Since z generates $C(\mathbb{T})$, $(\varphi_{\varepsilon+1})$ is unique.

Theorem 1.16. Suppose that w is an isometry in a unital C∗-algebra B. Then there exists a unique unital ∗-homomorphism.

$$\Phi: A \to B$$

Such that $\Phi\ (u) = w$. Moreover, if $ww^* \ \neq 1$ (so that w is notunitary), then $(\varphi_{\varepsilon+1})$ is isometric.

Proof: The uniqueness of Φ is evident by Remark 3.12. By building the GNS-construction, we have reduced to The case where B is a unital C^*-subalgebra of $B(H)$ containing id_H of some Hilbert space H. By the Weld-von Neumann decomposition, we can write.

$$\begin{cases} H = \bigoplus_{\lambda\in\Lambda} H_\lambda \\ w = \bigoplus_{\lambda\in\Lambda} w_\lambda \end{cases}$$

where H_λ are Hilbert spaces and each $w_\lambda \in B(H_\lambda)$ is a unitary or a unilateral shift.

If w_λ is unitary, then can fined by Lemma 3.15 composing all of the following maps

$$\mathbb{A} \overset{\pi}{\to} \mathbb{A}/K(H^2) \xrightarrow{(\psi_{\varepsilon+1})^{-1}} C(\mathbb{T}) \xrightarrow{(\varphi_{\varepsilon+1})} B(H_\lambda) \qquad (13)$$

where π is an quotient map, one gets a unital ∗-homomorphism $\Phi_\lambda: \mathbb{A} \ \to \ B\ (H_\lambda)$ such that $\Phi_\lambda\ (u) \ = \ w_\lambda$.

If w_λ is a one-sided transformation, then there is a monosyllabic $U_\lambda: H^2 \to H_\lambda$ such that $w_\lambda = U_\lambda\ uU_\lambda^*$. and then the map

$$\Phi_\lambda: \mathbb{A} \ \to \ B\ (H_\lambda),\ a \ \mapsto U_\lambda\ aU_\lambda^*. \qquad (14)$$

is an isometric unital ∗-homomorphism such that $\Phi_\lambda\ (u) \ = \ w_\lambda$.

Now, through the previous paragraphs, we get the family of representations $(H_\lambda, \Phi_\lambda)_{\lambda\in\Lambda}$ from C∗algabra $\mathbb{A}$. Make (H, Φ) the direct sum of $(H_\lambda, \Phi_\lambda)_{\lambda\in\Lambda}$. Then $\Phi: \mathbb{A} \to B\ (H)$ is a unital ∗-homomorphism such that $\Phi\ (u) = \bigoplus_\lambda w_\lambda = w$. Moreover, since $\Phi\ (u) \in B$ and u generates $\mathbb{A}$, one it follows that $\Phi\ (\mathbb{A}) \subseteq B$.

In the end, we assume that $ww^* \neq 1$. Then some w_{λ_0} is a unilateral shift. Hence, the representation $\left(H_{\lambda_0}, \Phi_{\lambda_0}\right)$ is faithful, then (H, Φ). Therefore, using the fact that the injective of ∗-homomorphism between C^*-algebras is necessarily isometric, Φ is isometric.

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