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The Set Multipartite Ramsey Numbers For Paths Versus Wheels

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Abstract – For graphs G_1 and G_2, the set multipartite Ramsey number $M_t(G_1, G_2) = j$ is the smallest integer such that every factorization of graph $K_{j\times t} := F_1 \oplus F_2$ satisfies the following we establish exact value of the set multipartite Ramsey number $M_t(P_n, W_s)$ for all integers condition: either F_1 contains G_1 as a subgraph or F_2 contains G_2 as a subgraph. In this paper, $t \geq 2$ where P_n is a path on n vertices and W_s is a wheel of order $s + 1$ vertices with $s \geq 3$, and $3 \geq n \geq 4$

Keywords – Cycle, Path, Set multipartite Ramsey number, Wheel.

I. INTRODUCTION

A graph G is said to be *factorable* into factors $G_1, \cdots, G_n$ if these factors are pairwise edge-disjoint and $\cup_{i=1}^{n} E(G_i) = E(G)$. If G is factored into $G_1, \cdots, G_n$, then $G := G_1 \oplus \cdots \oplus G_n$, which is called a *factorization* of G. Given a graph G and a subset H of vertices of G, the *induced subgraph* by H, denoted $G[H]$, is the graph composed of the vertices H and all edges in G among those vertices.

In [2], Burger and van Vuuren studied one of generalizations of the classical Ramsey number as follows. Let $K_{n\times l}$ denote a complete, balanced, multipartite graph consisting of n partite sets and l vertices per partite set. Let j, l, n, s and t be natural numbers with $n, s \geq 2$. The *set multipartite Ramsey number* $M_j(K_{n\times l}, K_{s\times t})$ is the smallest natural number ζ such that any red-blue coloring on the edges of $K_{\zeta\times j}$ necessarily forces a red $K_{n\times l}$ or blue $K_{s\times t}$ as a subgraph.

In this paper, we generalize this concept by releasing completeness requirement in the forbidden graphs as follows. Given two graphs G_1 and G_2, *set multipartite Ramsey number* $M_t(G_1, G_2) = j$, $(t, j \geq 2)$, is the smallest integer such that every factorization of graph $K_{j\times t} := F_1 \oplus F_2$ satisfies the following condition: either F_1 contains G_1 as a subgraph or F_2 contains G_2 as a subgraph.

Let P_s be a path with s vertices. A *cycle* C_n of length $n \geq 3$ is a connected graph on n vertices in which every vertex has degree two. A *wheel* W_n is a graph on $n + 1$ vertices obtained from a C_n by adding one vertex c, called the *hub* of the wheel, and making c adjacent to all vertices of the C_n, called the *rim* of the wheel. Define V_u to be a partite set of graph G containing the vertex u.

II. Small set numbers and some usefull theorems

There are only a few set multipartite Ramsey numbers know to the authors. These are $M_1(K_{2\times2}, K_{3\times3}) = 7$ due to Chartrand and Schuster [3], $M_1(K_{2\times2}, K_{4\times1}) = 10$ due to Chvátal and Harary [5], $M_2(K_{2\times2}, K_{3\times1}) = 4$ and $M_2(K_{2\times2}, K_{4\times1}) = 7$ due to Harborth and Mengersen [9,10], $M_1(K_{2\times2}, K_{5\times1}) = 14$ due to Greenwood and Gleason [4], $M_1(K_{2\times2}, K_{6\times1}) = 18$ due to Exoo [7], $M_1(K_{2\times3}, K_{2\times3}) = 18$ due to Harborth and Mengersen [8].

In [11], Effendi *et al.* studied size multipartite Ramsey theory for large paths versus wheel on four

vertices. For all $j \geq 2$, Burger and Vuuren [2] determined the exact values of $M_j(K_{2\times2}, H)$ where $H = K_{2\times2}$ or $K_{3\times1}$. Their results are presented in Table 1.

Table 1. The set multipartite Ramsey numbers for $(K_{2\times2}, K_{2\times2})$ and $(K_{2\times2}, K_{3\times1})$.

j	$M_j(K_{2\times2}, K_{2\times2})$	$M_j(K_{2\times2}, K_{3\times1})$
1	6[a]	7[b]
2	4[d]	4[c]
3	4[d]	3[e]
4	3[d]	3[e]
≥ 5	2[d]	3[e]

[a] Due to Chvátal and Harary [5]. [d] Due to Burger *et al.* [1].

[b] Due to Chartrand and Schuster [3]. [e] Due to Burger and Vuuren [2].

[c] Due to Harborth and Mengersen [9,10].

Theorem 2.1. (*Set numbers versus size numbers*).

1. *$m_k(K_{n\times l}, K_{s\times t}) > j$ if and only if $M_j(K_{n\times l}, K_{s\times t}) > k$, $\forall l, t \geq 1$ and $n, s \geq 2$.*
2. *$m_k(K_{n\times l}, K_{s\times t}) \leq j$ if and only if $M_j(K_{n\times l}, K_{s\times t}) \leq k$, $\forall l, t \geq 1$ and $n, s \geq 2$.*

In the proof of Theorem 1.5, we use the following Dirac's Theorem [6].

Theorem 2.2 (Dirac's Theorem, 1952). *A simple graph with n vertices ($n \geq 3$) is Hamiltonian if every vertex has degree at least $n\backslash 2$.*

The main results of this paper is determination of the set multipartite Ramsey number for paths on three and four vertices versus wheels.

III. Set Ramsey numbers related to P3 vs Wn

For any natural numbers s 2 and $t \geq$ clearly $K_{2\times j}$ contains no W_s. Therefore $M_j(G, W_s) = 2$ for any graph G. As a consequence, $M_j(P_n, W_s) = 2$ for $n = 3$ or $n = 4$. The following theorem determines the exact values of $M_j(P_n, W_s)$ for $j \geq 2$ and $n = 3$ or 4 where $G \cong P_n$, W_s with $s \geq 3$.

Theorem 3.1. *For positive integer $n \geq 3$,*

$$M_2(P_3, W_n) = \begin{cases} 4 & for\ n = 3 \\ \left\lceil \frac{n+3}{2} \right\rceil & for\ n \geq 4 \end{cases}$$

Proof.

We consider the following two cases.

Case 1: For $n = 3$.

We show first that $M_2(P_3, W_3) \geq 4$, we consider $F = K_{3\times 2}$. Suppose $U_i = \{u_{i1}, u_{i2}\}$ is the partite sets in graph, where $i = 1, 2, 3$. Let $F_1 \oplus F_2$ be any factorization of F and suppose that P_3 is not $M^1 = \{u_{11}u_{21}, u_{12}u_{31}, u_{22}u_{32}\}$. Assume that u_{11} is the hub candidat of wheel in F_2. Suppose a subgraph of F_1. Without loss of generality, we may assume that F_1 induced of perfect matching $A \cong V(F)\backslash(V_1 \cup \{u_{21}\})$. Since $F_2[A]$ only consist of one edge $u_{22}u_{31}$ and one vertex u_{32} then F_2 contains no cycle C_3. As a consequence, F_2 contains no wheel W_3. Therefore $M_2(P_3, W_3) \geq 4$.

We show upper bound $M_2(P_3, W_3) \leq 4$. Let $G_1 \oplus G_2$ be any factorization of $G = K_{4\times 2}$ such that $G_1 \not\supseteq P_3$. We will show that $G_2 \supseteq W_3$. Let $V_i = \{v_{i,1}, v_{i,2}\}$ with $i = 1, 2, 3, 4$ the partite sets in G. Without loss of generality, we may assume that F_1 induced of perfect matching $M^2 = \{v_{11}y_{21}, v_{12}y_{41}, v_{22}x_{31}, v_{32}v_{42}\}$. Suppose $v_{1,1}$ is hub of wheel W_3 in F_2. We may assume that $P := v_{21}v_{22}v_{31}v_{32}v_{42}$ is a path in F_1. Suppose $B \cong V(G)\backslash(V_1 \cup \{x_{21}\})$. Since $G_1 \not\supseteq P_3$ then $\Delta(G_1) \leq 1$. So, $\delta(G_2[B]) \geq 3$. By Theorem 2 in [**6**], G_2 contain a cycle on n vertices with $3 \leq n \leq 6$. Hence $G_2 \supseteq W_3$. Therefore $M_2(P_3, W_3) \leq 4$.

Case 2: For $n \geq 4$.

Suppose $j = \left\lceil \frac{n+3}{2} \right\rceil$. We will show first that $M_2(P_3, W_n) \geq j$ for $n \geq 4$. Suppose $F = K_{(j-1)\times 2}$. Consider the factorization $F_1 \oplus F_2$ of F such that F_1 is a perfect matching. Without loss of generality, we assume that $M = \{x_{11}x_{21}, x_{12}x_{31}, x_{22}x_{32}\}$ is a perfect matching in F_1, and suppose x_{11} is hub of wheel W_n in F_2. Suppose $C \cong V(F)\backslash (V_1 \cup \{x_{21}\})$. Since $F[C]$ only consist of one edge and one vertex in F_2, then F_2 contains no cycle. As a consequence, F_2 contains no wheel W_n. Therefore $M_2(P_3, W_n) \geq j$ for $n \geq 4$.

To show that upper bound $M_2(P_3, W_n) \leq j$ for $n \geq 4$. Let $G_1 \oplus G_2$ be any factorization of $G = K_{j\times 2}$ such that $G_1 \not\supseteq P_3$. We will show that $G_2 \supseteq W_n$ with $n \geq 4$. Let $V_i = \{v_{i1}, v_{i2}\}$ with $i = 1, 2, 3, 4$ be a partite sets of G. Since $G_1 \not\supseteq P_3$ then $\Delta(G_1) \leq 1$. Since $\Delta(G) = 2(j-1)$ and $\Delta(G_1) \leq 1$, then $\delta(G_2) \geq 2(j-1) - 1$. Thus, by Dirac's Theorem [6], G_2 possesses a hamiltonian cycle (of order $n = 2(j-1) - 1$), so C_n is a subgraph of F_2. As a consequence $F_2 \supseteq W_n$ with $n \geq 4$. therefore $M_2(P_3, W_3) \leq j$ for $n \geq 4$.

IV. SET RAMSEY NUMBERS RELATED TO P4 VS WN

The following theorem, we will determine set multipartite Ramsey numbers for path four vertices versus wheel on n vertices.

Theorem 4.1. *For integers $n \geq 3$,*

Proof.

$$M_2(P_4, W_n) = \begin{cases} 5 & for\ 3 \le n \le 7 \\ \frac{n}{2} + 2 & for\ n \ge 8\ even \\ \left\lceil \frac{n}{2} \right\rceil + 2 & for\ n \ge 9\ odd\ and\ n \equiv 0\ mod\ 3 \\ \frac{n+1}{2} + 1 & for\ n \ge 11\ odd\ and\ n\ \not\equiv 0\ mod\ 3 \end{cases}$$

To show lower bound, we consider as follow. For $3 \le n \le 7$, we show that $M_2(P_4, W_n) \ge 5$. Let $F_1 \oplus F_2$ be a factorization of $F \cong K_{4\times 2}$ with $F_1 = 2C_3 \cup P_2$ and F_2 is a complement of F_1 respect to F . Clearly that $F_1 \not\supseteq P_4$ and $F_2 \not\supseteq C_n$ for $3 \le n \le 7$. As a consequence, $F_2 \not\supseteq W_n$ for $3 \le n \le 7$. Therefore $M_2(P_4, W_n) \ge 5$ for $n \in \{3, 4, 5, 6, 7\}$.

Suppose $j \in \{\frac{n}{2} + 2$ for $n \ge 8$ even, $\left\lceil \frac{n}{2} \right\rceil + 2$ for $n \ge 9$ odd and $n \equiv 0$ mod 3, $\frac{n+1}{2} + 1$ for $n \ge 9$ odd and $n \equiv 0$ mod 3 . Consider factorization of $F \cong K_{(j-1)\times 2} = F_1 \oplus F_2$ with $F_1 \not\supseteq P_4$. Since form a wheel in F_2 then we need one vertex c as hub of partite set in F_2 and also F_1 contains no P_4, then $2(j-2)-|\{c\}| = 2(j-2) - 1 < n$. Thus, F_2 contains no wheel W_n. Therefore $M_2(P_4, W_n) \ge j$ on n vertices.

Next, We will show upper bound $M_2(P_4, W_n) \le j$, let $G_1 \oplus G_2$ be any factorization $G = K_{j\times 2}$ such that $G_1 \not\supseteq P_4$. To show that $G_2 \supseteq W_n$, we consider two possiblities.

First if either there are two vertex $x, y \in V(G_1)$ with $d(x) = d(y) = t$-1 or $d(x) < t$-1 for every vertex $x \in V(G_1)$, then $\delta(G_2) \ge j$-1. Since $G_1 \not\supseteq P_4$, then there is one vertex $c \in V(G_1)$ such that $d(c) \le 2$. Let A be a neighborhood of a vertex c. Suppose $c \in H = V(G_1) \backslash (V_c \cup A)$. Thus, by Dirac's Theorem [6], G_2 possesses a hamiltonian cycle (of order $n \ge 2(j-1)-2$). Together with c, the cycle C_n in $G_2[H]$ will form a W_n in G_2. As a consequence, G_2 also contains a W_n in G_2. Therefore $M_2(P_4, W_n) \le j$.

The second case, there is at least one vertex $x \in V(F_1)$ such that $d(x) > j$-1. Choose $c \in V(G_1)$ which $cx \in V(G_1)$, so clearly that $cu \notin E(G_1)$ for every vertex u in $V(G_1)$. Consider $H = V(F_2)$ $(V_c \cup \{x\}$). Since $d(u) \ge |V(H)| / 2$ for every $u \in V(H)$, then by Dirac's Theorem [6], $G_2[H]$ contains hamiltonian cycle C_n. Together with x, the cycle C_n in $G_2[H]$ will form a W_n in G_2. As a consequence, G_2 also contains a W_n in G_2. Therefore $M_2(P_4, W_n) \le j$. This completes the proof of the theorem.

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