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Variational Principle for Traveling Waves of a Generalized Benjamin-Bona-Mahony Equation using Semi-Inverse Method

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Abstract – The semi-inverse method was claimed as an extremely simple method in constructing the variational principles for a wide range of nonlinear problems. In this paper, we apply the method to obtain the variational principle for traveling waves of a generalized Benjamin-Bona-Mahony equation. From the performed calculations, we confirm the effectiveness of the method.

Keywords – Benjamin-Bona-Mahony Equation, Variational Principle, Semi-Inverse Method.

I. Introduction

One of the features in nonlinear partial differential equations, which plays an important role in physics and mathematics, is its corresponding variational principles. This is because the variational principles give a basis for developing a variety of analytical and numerical approximate methods to the equations [1]. Mostly the researchers obtained the variational principles by Noether's theorem (see, e.g., [2]). The derivation is clear but rather heavy and tedious in calculations. One of the alternative techniques is the so-called semi-inverse method which developed by He [3]. Due to its simplicity and concise result, the method has been widely used and claimed as a promising technique to construct the variational principles (see, e.g., [4, 5, 6]).

In the present study, we consider a generalized Benjamin-Bona-Mahony equation which is written as

$$u_t + u_x + uu_x - \alpha u_{xt} - u_{xxt} = 0. \tag{1}$$

The variational principle for traveling waves of Eq. (1) is obtained by employing the semi-inverse method developed by He [3]. When $\alpha = 0$, Eq. (1) is known as a Benjamin-Bona-Mahony (BBM) equation in 'standard' form. The BBM equation was firstly introduced by Brooke Benjamin, Jerry L. Bona and John Joseph Mahony in 1972 to model the surface water waves propagating in one direction for any wave number [7]. Without loss of generality, here we consider the case $\alpha = 1$.

II. Implementing the Semi-Inverse Method

To seek traveling wave solutions of Eq. (1), we substitute the transformation

$$u(x,t) = v(\xi), \quad \xi = x + st, \tag{2}$$

where s denotes wave velocity, into Eq. (1), so that we obtain

$$(s+1)v' + vv' - sv'' - sv^{(3)} = 0, \tag{3}$$

where the prime denotes the derivative with respect to ξ. Performing integration of Eq (3) with respect to ξ, one yields

$$(s+1)v \quad + \frac{1}{2}v^2 - sv' - sv'' = 0, \tag{3}$$

where the resulting integration constant has been taken to be zero.

Firstly, let us consider the simple case as follows:

$$(s+1)v \quad + \tfrac{1}{2}v^2 - sv'' = 0. \tag{4}$$

It is easy to show that the variational principle for Eq. (4) is

$$\tilde{L}(v) = \int \frac{s+1}{2}v^2 + \frac{1}{6}v^3 + \frac{1}{2}s(v')^2 d\xi. \tag{5}$$

Secondly, by following the semi-inverse method [3], we introduce an integrating factor $f(\xi)$, which is an unknown function of wave variable ξ, and then consider the following integral

$$L(v) = \int f(\xi)\left[\frac{s+1}{2}v^2 + \frac{1}{6}v^3 + \frac{1}{2}s(v')^2\right] + F(v, v', v'', \dots)d\xi, \tag{6}$$

where F is an unknown function of v and/or its derivatives. The Euler–Lagrange equation of Eq. (6) then gives

$$(s+1)fv + \frac{1}{2}fv^2 - sf'v' - sfv'' + \frac{\delta F}{\delta v} = 0, \tag{7}$$

where $\frac{\delta F}{\delta v}$ is called variational derivative which is defined as

$$\frac{\delta F}{\delta v} = \frac{\partial F}{\partial v} - \frac{d}{d\xi}\frac{\partial F}{\partial v'} + \frac{d^2}{d\xi^2}\frac{\partial F}{\partial v''} + \cdots.$$

Next, by dividing each terms in Eq. (7) by f, we have

$$(s+1)v + \frac{1}{2}v^2 - \frac{sf'}{f}v' - sv'' + \frac{\delta F}{f\delta v} = 0. \tag{8}$$

Comparing the expression between Eq. (8) and Eq. (3), we conclude

$$\frac{f'}{f} = 1 \text{ and } \frac{\delta F}{\delta v} = 0. \tag{9}$$

Solutions of Eqs. (9) are given respectively by

$$f = e^{\xi} \text{ and } F = 0. \tag{10}$$

From the above results, we finally obtain the variational principle for Eq. (3) as follows

$$L(v) = \int e^{\xi}\left[\frac{s+1}{2}v^2 + \frac{1}{6}v^3 + \frac{1}{2}s(v')^2\right]d\xi. \tag{11}$$

III. Conclusion

The semi-inverse method has been implemented to construct the variational principle for traveling waves of a generalized Benjamin-Bona-Mahony equation (1). From the calculations performed in this paper, we confirm the extreme simplicity of the method.

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