International Journal of Progressive Sciences and Technologies (IJPSAT)
ISSN: 2509-0119.

http://ijpsat.ijsht-journals.org
Vol. 25 No. 1 February 2021, pp. 379-382

On A Cauchy Problem In A Hilbert Space With Operator Coefficients

Gafarov Ilg'or Ahmedjanovich[1] and Eshmatov Davron Abduvaxobovich[2]

[1]Namangan Engineering Construction Institute,
Namangan, Republic of Uzbekistan

[2]Namangan Engineering Construction Institute,
Namangan, Republic of Uzbekistan

Abstract – This article provides feedback on the operator coefficients of the Cauchy problem in the Hilbert phase.

Keywords – Function Of Scalar Argument, Generating Vector, Inverse Differentiation.

Consider the following Cauchy problem in Hilbert space H

$$\frac{d^n u(t)}{dt^n}+\sum_{k=1}^{n} A_k(t)\frac{d^{n-k}u(t)}{dt^{n-k}}=f(t), \qquad (1)$$

$$\left.\frac{d^k u(t)}{dt^k}\right|_{t=0}=u^{(k)}(0), \quad k=0,1,...,k-1, \qquad (2)$$

Where $u(t),\ f(t)-$ scalar argument functions $t\in[0,T]$ with values in $D(A)=\cap D(A_k(t))\subset H$, $A_k(t)-$ generally speaking, linear unbounded operators acting in H. As is known [1], such problems are usually ill-posed in the classical sense. Therefore, we will investigate for conditional correctness.

Previously, such a problem for "l-correctness" was considered in [2]. In this work, the conditional stability is estimated in another way.

Let be H - separable Hilbert space. Let us recall some concepts that we will use in what follows.

The decomposition of unity is a one-parameter family of projecting operators E_t, given in a finite or infinite interval $[\alpha,\beta]$ and satisfying the following conditions:

$a)\ \ E_\lambda E_\mu = E_s \quad (S=\min\{\lambda,\mu\});$

$b)$ in the sense of strong convergence $E_{\lambda-0}=E_\lambda \quad (\alpha<\lambda<\beta);$

$c)\ \ E_\alpha=0,\ E_\beta=I.$

Having an interval $\Delta=[\lambda^{'},\lambda^{"}]\subset[\alpha,\beta]$, we will be the difference $E_{\lambda^{"}}-E_{\lambda^{'}}$ denote E_Δ.

The spectrum of a self-adjoint operator is called simple if there is such a vector $g \in H$ (generating vector), which is dense in H linear span of a set of vectors $E(\Delta)g$, Where Δ runs through the set of all intervals on the number axis.

Let the operators $A_k(t)$ in equation (1) have strongly continuous derivatives up to the order $n-k$ $(k=1,2,...,n)$ inclusive. To equation (1) we apply n times the inverse differentiation operator

$$u(t)+\frac{1}{(n-1)!}\left\{\int_0^t (t-\tau)^{n-1} A_1(\tau)u^{(n-1)}(\tau)d\tau+...+\int_0^t (t-\tau)^{n-1} A_n(\tau)u(\tau)d\tau\right\}=$$

$$=\frac{1}{(n-1)!}\int_0^t (t-\tau)^{n-1}\cdot f(\tau)d\tau+\sum_{k=0}^{n-1}\frac{t^k}{k!}u^{(k)}(0).$$

In each integral of the last equation, the derivatives $u^{(k)}(t)$ we transfer to other factors. Then we get

$$u(t)+(Nu)(t)=\varphi(t), \qquad (3)$$

Where

$$(Nu)(t)=\int_0^t N_1(t,\tau)u(\tau)d\tau,$$

$$N_1(t,\tau)=\frac{1}{(n-1)!}\sum_{m=0}^{n-1}\sum_{k=0}^{m}(-1)^{k+m-1}\frac{C_m^k}{(n-k-1)!}(t-\tau)^{n-k-1}A_{n-m}^{(m-k)}(\tau),$$

$$\varphi(t)=\frac{1}{(n-1)!}\int_0^t (t-\tau)^{n-1} f(\tau)d\tau+\sum_{k=0}^{n-1}\frac{t^k}{k!}u^{(k)}(0)+$$

$$+\sum_{l=1}^{n-1}\sum_{k=1}^{n-l}\sum_{m=1}^{k}(-1)^m\frac{t^{n+m-k-1}}{(n+m-k-1)!}A_l^{(m-1)}(0)u^{(n-l-k)}(0).$$

Suppose now that there is a constant self-adjoint completely continuous operator with simple spectrum such that

$$\|S\|<1, \quad \|SN_1\|\le M, \quad \|SN\|=\|NS\|\le 1. \qquad (4)$$

By condition $S=\int f(\lambda)dE_\lambda$ and $u(t)=\int u(\lambda,t)dE_\lambda g$ in $D(S)$, Where $g-$ generating vector.

We will assume that $|f(\lambda)|\ge\lambda^{-1}$, then the following holds.

Theorem. Let be $\|\varphi(t)\|\le\varepsilon$ and for some $a,c>0$

$$|u(\lambda,t)|\le c\cdot\lambda^{-a}.$$

Then

$$\|u(t)\|\le 2c^{\frac{k_\varepsilon}{k_\varepsilon+a}}\cdot(2k_\varepsilon\cdot\varepsilon)^{\frac{a}{k_\varepsilon+a}},$$

Where k_ε equation solution $(k!)^{-1}=k\cdot\varepsilon$.

Evidence. From equation (3) after applying the operator S, get

$$S\,u(t)=A(t)\cdot u(t)+S\,\varphi(t), \qquad (5)$$

Where $A(t)=N(t)\cdot S$.

We apply to equation (5) the operator S:

$$S^2 u(t)=S A(t)u(t)+S^2 \varphi(t).$$

Hence, taking into account (5)

$$S^2 u(t)=A(t)S u(t)+S^2 \varphi(t)=A^2(t)u(t)+[A(t)S+S^2]\varphi(t).$$

To the last equality we apply again the operator S

$$S^3u(t)=A^2(t)S u(t)+[A(t)S^2+S^3]\varphi(t)=$$
$$=A^3(t)u(t)+[A^2(t)S+A(t)S^2+S^3]\varphi(t).$$

By doing the same k once we arrive at the following equation

$$S^k u(t)=A^k(t)u(t)+\left[\sum_{l=1}^{k} A^{k-l}(t)S^l\right]\varphi(t).$$

Hence, using (4), we obtain

$$\left\|S^k u(t)\right\|\le\left\|A^k(t)\right\|\cdot\left\|u(t)\right\|+k\cdot\varepsilon \qquad (6)$$

Applying the Cauchy formula, taking into account (4), it is easy to verify that

$$\left\|A^k(t)\right\|\le(Mt)^k\cdot(k!)^{-1}\le(k!)^{-1}.$$

Now if c is such that $\left\|u(t)\right\|\le 1$, from (6) it follows

$$\left\|S^k u(t)\right\|\le 2k\varepsilon. \qquad (7)$$

Hence we conclude for a fixed λ_0

$$\int_{|\lambda|<\lambda_0}\left|u(\lambda,t)\right|^2(dE_\lambda g,g)\le 4k^2\varepsilon^2\cdot\lambda_0^{2k}. \qquad (8)$$

Really,

$$\lambda_0^{-2k}\int_{|\lambda|\le\lambda_0}\left|u(\lambda,t)\right|^2(dE_\lambda g,g)\le\int_{|\lambda|\le\lambda_0}|\lambda|^{-2k}\left|u(\lambda,t)\right|^2(dE_\lambda g,g)\le$$
$$\le\int\lambda^{-2k}\left|u(\lambda,t)\right|^2(dE_\lambda g,g)\le\int\left|f(\lambda)\right|^{2k}\left|u(\lambda,t)\right|^2(dE_\lambda g,g)\le 4k^2\varepsilon^2.$$

On the other hand

$$\int_{|\lambda|>\lambda_0}\left|u(\lambda,t)\right|^2(dE_\lambda g,g)\le c^2\int_{|\lambda|>\lambda_0}|\lambda|^{-2a}(dE_\lambda g,g)\le c^2\cdot|\lambda|_0^{-2a}\int(dE_\lambda g,g)\le c^2\lambda_0^{-2a}.$$

From here and from inequality (8) we obtain:

$$\left\|u(t)\right\|^2=\int\left|u(\lambda,t)\right|^2(dE_\lambda g,g)=\left(\int_{|\lambda|\le\lambda_0}+\int_{|\lambda|>\lambda_0}\right)\left|u(\lambda,t)\right|^2(dE_\lambda g,g)\le 4k^2\varepsilon^2\lambda_0^{2k}+c^2\lambda_0^{-2a}.$$

Finding the optimal λ_0 (equating the terms on the right-hand side of the last inequality) we obtain the statement of the theorem.

Comment. It should be noted that from the equation $(k!)^{-1}=k\cdot\varepsilon$ follows that $k_\varepsilon\to\infty$

at $\varepsilon \to 0.$ Hence, it is easy to see that

$$c^{\frac{k_\varepsilon}{k_\varepsilon + a}} \cdot \left(2k_\varepsilon \cdot \varepsilon\right)^{\frac{a}{k_\varepsilon + a}} \to 0 \text{ at } \varepsilon \to 0. \quad (*)$$

REFERENCES

[1] Lavrent'ev M.M., Saveliev L.A. Operator theory and ill-posed problems. –Novosibirsk: Publishing House of the Institute of Mathematics. 1999.-702s.

[2] I. A. Gafarov. On One Conditionally Correct Cauchy Problem. International journal of progressive sciences and technologies. Vol. 24, № 1 (2020): December. P.P. 463-468.