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ZZ Transform Method for Natural Growth and Decay Problems

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Abstract — **The new integral transform "ZZ Transform Method'' was introduced by Zain Ul Abadin Zafar [1] in 2016. In this paper, I applied this new integral transform technique for solving first order linear differential equation, obtained in the problems based on a property either Law of Natural Growth or Law of Natural Decay. ZZ Transform Method is similar to the other transforms such as Laplace and Sumudu Transforms. The advantage of ZZ Transform Method is that it solves a differential equation with variable coefficients.**

Keywords — **ZZ Transform Method; Law of Natural growth and decay; Differential equations.**

I. Introduction

In science and engineering Differential equations plays an important role. In order to solve the certain differential equations, the integral transforms were extensively used. The importance of an Integral Transforms is that they provide powerful operational methods for solving initial value problems. Mostly we use Laplace Transform technique for solving differential equations. ZZ Transform Method is the new integral transform method which is introduced by Zain Ul Abadin Zafar [1], is very much useful to solve a differential equation with variable coefficients.

II. Definition Of Zz Transform

Let f(t) be a function defined for all $t\geq 0$. The ZZ transform of f(t) is the function Z(v,s) defined by

$$Z(v,s) = H\{f(t)\} = s\int_0^\infty f(vt)e^{-st}dt \quad \_\_(a)$$

provided the integral exists. Equation (1.1.1) can be written as

$$Z(v,s) = H\{f(t)\} = \frac{s}{v}\int_0^\infty f(t)e^{\frac{-st}{v}}dt \quad \_\_\_(b)$$

III. Existence Of Zz Transform Method

If f(t) is piecewise continuous in every finite interval $0\leq t\leq k$ and of exponential order ρ for t>k, then its ZZ transform Z(v,s) exists for al s>ρ, v>ρ.[2]

IV. Iv. Zz Transform Of Some Required Functions, Properites

(i) Let $f(t)=e^{at}$, where $t\geq 0$ and a is a constant, then $H\{e^{at}\} = \dfrac{s}{s-va}$

(ii) Let $f(t) = \begin{cases} 1 & t>0 \\ 0 & t\leq 0 \end{cases}$ then $H\{1\} = 1$

(iii) Let $f(t)=t^n$, where $n\in N$, then $H\{t^n\} = n!\dfrac{v^n}{s^n}$

(iv) (Linear Property) If a, b are constants and f(t) and g(t) are functions, then $H\{a\,f(t)+b\,g(t)\} = a\,H\{f(t)\}+b\,H\{f(t)\}$

(v) (ZZ Transform for Derivatives)

Let $H\{f(t)\}=Z(v,s)$, then

$$H\left\{f^{1}(t)\right\}=\frac{s}{v}Z(v,s)-\frac{S}{v}f(0)$$

V. Law Of Natural Growth And Decay

Let x(t) be the amount of substance at time 't' and it is getting converted chemically. A law of chemical conversion states that the rate of change of amount x(t) is proportional to the amount of the substance available at that time 't'.

i.e. $\frac{dx}{dt}\ \alpha\ x \Rightarrow \frac{dx}{dt}=k\ x$ where 'k>0' is the rate constant. This is a first order first degree differential equation which represents natural growth.

Similarly, the differential equation for natural decay is $\frac{dx}{dt}=-k\ x$ where 'k>0' is the rate constant.

VI. Application Of ZZ Transform Method

A. The rate at which bacteria multiply is proportional to the instantaneous number present. If the original number is doubles in 2 hours, in how many hours will it be triple?

Solution: Let N represent the number of bacteria at any time t. At the initial time t=0, N=x.

From Law of natural growth, we have

$$\frac{dN(t)}{dt}\propto N(t)\Rightarrow\frac{dN(t)}{dt}=k\,N(t)$$
$$where\ k\ is\ constant$$

$$\Rightarrow N^{1}(t)=kN(t)\ or\ N^{1}(t)-kN(t)=0\ \_\_(1)$$

Now taking ZZ Transform both sides of (1),

$$H\left\{N^{1}(t)-k\,N(t)\right\}=H\{0\}$$

$$\Rightarrow H\left\{N^{1}(t)\right\}-H\{k\,N(t)\}=0$$

$$\Rightarrow\left[\frac{s}{v}H\{N(t)\}-\frac{s}{v}N(0)\right]-k\,H\{N(t)\}=0$$

$$\Rightarrow H\{N(t)\}\left(\frac{s}{v}-k\right)=\frac{s\,x}{v}\qquad(\because N(0)=x)$$

$$\Rightarrow H\{N(t)\}=\left(\frac{s\,x}{v}\right)\left(\frac{v}{s-kv}\right)$$

$$\Rightarrow H\{N(t)\}=\left(\frac{sx}{s-kv}\right)$$

Taking inverse ZZ transform, then we get

$$N(t)=xe^{kt}\qquad\_\_\_\_\_\_\_\_\_\_(2)$$

Given that at t=2hours, N=2x, From (2.2.1),

$$2x=xe^{2k}\Rightarrow 2=e^{2k}\Rightarrow e^{k}=2^{\frac{1}{2}}\ \_\_\_\_(3)$$

If N=3x, from (2.2.1) we get

$$3x=xe^{kt}\Rightarrow 3=e^{kt}\Rightarrow 3=\left(e^{k}\right)^{t}\Rightarrow 3=(2)^{t/2}$$

Taking both sides logarithm, then we have

$$\log 3=\frac{t}{2}\log 2\ \Rightarrow t=\frac{2\log 3}{\log 2}hours.$$

B. The number N of bacteria on a culture grew at a rate proportional to N. The value of N was initially 100 and increased to 332 in one hour. What was the value of N after 90 minutes?

Solution: Given, at t=0, N=100. At t=60minutes, N=332. Then at t=90minutes, N=?

From law of Natural growth, we have equation (1).
Now taking ZZ Transform on both sides (1), we get

$$H\left\{N^{1}(t)-k\,N(t)\right\}=H\{0\}$$

$$\Rightarrow H\left\{N^{1}(t)\right\}-H\{k\,N(t)\}=0$$

$$\Rightarrow\left[\frac{s}{v}H\{N(t)\}-\frac{s}{v}N(0)\right]-k\,H\{N(t)\}=0$$

Applying the initial condition N(0)=100, then we have

$$\Rightarrow H\{N(t)\}\left(\frac{s}{v}-k\right)=\frac{100\,s}{v}\qquad(\because N(0)=100)$$

$$\Rightarrow H\{N(t)\}=\left(\frac{s100}{v}\right)\left(\frac{v}{s-kv}\right)$$

$$\Rightarrow H\{N(t)\}=100\left(\frac{s}{s-kv}\right)$$

Taking inverse ZZ Transform on both sides, we get

$$\Rightarrow N(t)=100\,H^{-1}\left\{\frac{s}{s-vk}\right\}\Rightarrow N(t)=100e^{kt}\ \_\_\_\_(4)$$

At t=60min, N(t)=332. From (4), we have

$$332=100e^{60k}\Rightarrow e^{k}=\left(\frac{332}{100}\right)^{\frac{1}{60}}\qquad\_\_\_\_\_\_\_\_\_\_\_\_(5)$$

Now to find N(t) at t=90minutes, From(5), we have $N(t)=100e^{90k}=100(e^{k})^{90}$ ____________(6)

Substituting (5) in (6), we get

$$N(t)=100\left(\frac{332}{100}\right)^{\frac{90}{60}}=604.93\quad\Rightarrow N(t)\cong 605$$

Therefore the number of bacteria after 90 minutes is 605.

C. Example: If 30% radioactive disappear in 10days then how long it will take to disappears 90%?

Solution: Let x(t) be the amount of substance at any time t. Given that at time t=0, x=100%=100. Also, given that at time t=10days, x=70%=70. Now we have to find the time required to become x=10%=10.

From Law of natural decay, we have

$$\frac{d\,x(t)}{dt} = -k\,x(t) \Rightarrow \quad x^1(t) + k\,x(t) = 0 \text{ \_\_\_\_\_\_\_\_\_\_\_(7)}$$

where 'k' is the rate constant.

Now taking ZZ Transform both sides,

$$H\left\{x^1(t) + k\,x(t)\right\} = H\,\{0\}$$

$$\Rightarrow H\left\{x^1(t)\right\} + H\,\{k\,x(t)\} = 0$$

$$\Rightarrow \left[\frac{s}{v} H\,\{x(t)\} - \frac{s}{v} x(0)\right] + k\,H\,\{x(t)\} = 0$$

Applying the initial condition at t=0, x(t)=100.

$$\Rightarrow H\,\{x(t)\}\left(\frac{s}{v} + k\right) = \frac{100\,s}{v} \qquad (\because x(0) = 100)$$

$$\Rightarrow H\,\{x(t)\} = \left(\frac{s100}{v}\right)\left(\frac{v}{s+kv}\right)$$

$$\Rightarrow H\,\{x(t)\} = 100\left(\frac{s}{s+kv}\right)$$

Taking inverse ZZ Transform on either side,

$$\Rightarrow x(t) = 100\ H^{-1}\left\{\frac{s}{(s+kv)}\right\}$$

$$\Rightarrow x(t) = 100\ e^{-kt} \text{ \_\_\_\_\_\_\_\_\_\_\_(7)}$$

Also, given that at t=10 days, x (t) =70. From (7), we get

$$70 = 100e^{-10k} \Rightarrow 0.7 = e^{-10k} \Rightarrow e^{-k} = (0.7)^{\frac{1}{10}}$$

If x (t) =10, from (7),

$$10 = 100e^{-kt} \Rightarrow \ 0.1 = (e^{-k})^t = (0.7)^{t/10} \text{ \_\_\_(8)}$$

Taking the natural logarithm on both sides, then we have

$$\log_e(0.1) = \frac{t}{10}\log_e(0.7)$$

$$\Rightarrow t = \frac{10\log_e(0.1)}{\log_e(0.7)} \approx 64.557\,days$$

D. The rate at which a certain substance decomposes in a certain solution at any instant is proportional to the amount of it present in the solution at that instant. Initially, there are 27 grams and three hours later, it is found that 8 grams are left. How much substance will be left one more hour?

Solution: Let 'm' grams is the amount of the substance left in the solution at any time 't'.Using natural decay principle, we get the equation (7).

Now taking ZZ Transform both sides, then we have

$$H\left\{x^1(t) + k\,x(t)\right\} = H\,\{0\}$$

$$\Rightarrow H\left\{x^1(t)\right\} + H\,\{k\,x(t)\} = 0$$

$$\Rightarrow \left[\frac{s}{v} H\,\{x(t)\} - \frac{s}{v} x(0)\right] + k\,H\,\{x(t)\} = 0$$

Applying the initial condition at t=0, x(t)=27 grams.

$$\Rightarrow H\{x(t)\}\left(\frac{s}{v} + k\right) = \frac{27\,s}{v} \qquad (\because x(0) = 27)$$

$$\Rightarrow H\,\{x(t)\} = \left(\frac{27\,s}{v}\right)\left(\frac{v}{s+kv}\right)$$

$$\Rightarrow H\{x(t)\} = 27\left(\frac{s}{s+kv}\right)$$

Taking inverse ZZ Transform on either side,

$$\Rightarrow x(t) = 27\ H^{-1}\left\{\frac{s}{(s+kv)}\right\}$$

$$\Rightarrow x(t) = 27\ e^{-kt} \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_(9)}$$

Using the another given condition, at t=3hours,

x (t) =8grams, from (9) we have

$$8 = 27e^{-3k} \Rightarrow e^{-3k} = \frac{8}{27} \Rightarrow e^{-k} = \left(\frac{8}{27}\right)^{\frac{1}{3}} \text{ \_\_\_(10)}$$

Now to find x(t) after the time t=4hours, from (2.4.1), we get

$$x(t) = 27e^{-4k} \Rightarrow x(t) = 27\left(e^{-k}\right)^4$$

$$\Rightarrow x(t) = 27\left(\frac{8}{27}\right)^{\frac{4}{3}} \Rightarrow x(t) = \frac{16}{3}\,grams$$

Therefore

uantity of the after the time t=4 hours is 5.333 grams.

VII. CONCLUSION:

We can apply ZZ Transform Method for solving the problems connected to natural growth and decay.

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