International Journal of Progressive Sciences and Technologies (IJPSAT)
ISSN: 2509-0119.

http://ijpsat.ijsht-journals.org
Vol. 17 No. 1 October 2019, pp. 51-54

Variational Principle for a Non-Autonomous Cubic-Quintic Discrete Nonlinear Schrödinger Equation

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Abstract - We apply the semi-inverse method to formulate variational principle for a non-autonomous cubic-quintic discrete nonlinear Schrödinger equation. The equation can describe the optical beam propagation in coupled nonlinear waveguides whose material is better modeled by competing cubic and quintic nonlinearities. Our formulation is performed by introducing integrating factors and an unknown function into the existing variational principle of the corresponding autonomous version. From the obtained result we confirm the effectiveness of the method.

Keywords - Cubic-Quintic Discrete Nonlinear Schrödinger Equation, Variational Principle, Semi-Inverse Method.

I. Introduction

The discrete nonlinear Schrödinger (DNLS) equation is one of the most fundamental nonlinear discrete models, because it describes many important phenomena in various applications, such as dynamics of nonharmonic coupled oscillator systems, optical beam propagation in coupled nonlinear waveguides, and dynamics of Bose-Einstein condensation (BEC) [1]. There are many variations of this equation and they are often named based on its nonlinearity. For example, the DNLS equation with cubic and quintic nonlinearities is called as cubic-quintic DNLS equation and is expressed as [2] :

$$i\dot{\psi}_n = -C(\psi_{n-1} - 2\psi_n + \psi_{n+1}) - B|\psi_n|^2\psi_n + Q|\psi_n|^4\psi_n, \tag{1}$$

Where $\psi_n \equiv \psi_n(t) \in \mathbb{C}$ is a wave function at time $t \in \mathbb{R}^+$ and site $n \in \mathbb{Z}$, $\dot{\psi}_n$ represents the derivative of the function ψ_n with respect to t, and B and Q measure the cubic and quintic nonlinearity strengths, respectively.

In the context of application in optical beam of coupled nonlinear waveguides, the quantity $|\psi_n|^2$ in Eq. (1) represents the intensity of optical field. When $Q = 0$, which corresponds to a medium with a Kerr nonlinearity, Eq. (1) becomes a cubic DNLS equation and is very much studied in many literatures (see, e.g., [1] and the references therein). However, recent experiment [3] has confirmed that the nonlinear effect on some waveguide material is better modeled by the addition of competing quintic nonlinearity, i.e. when $B > 0$ and $Q > 0$.

One of the analytical approaches which is often used to investigate the existence and stability of solutions of the DNLS equations is the so-called variational approximation method. This method is developed based on the principle of the least action which states that the equation of motion is determined by the critical points of its action, i.e. the

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integral or sum of Lagrangian with respect to time [4]. The Lagrangian, or also known as variational principle, plays an important role in both physics and mathematics. It has been mostly formulated by the Noether's theorem (see, e.g., [5]), but its derivation is rather cumbersome mathematically. As an alternative method, He has developed the so-called semi-inverse method [6] to construct variational principle for continuous system with simple and effective calculations. Many researchers have used the He's method in various sytems (see, e.g., [7, 8, 9, 10]).

In this paper, we apply for the first time, to the best of our knowledge, the semi-inverse method for establishing variational principle for discrete system, by taking example in a non-autonomous cubic-quintic discrete nonlinear Schrödinger (QC-DNLS) equation in the form :

$$it\dot{\psi}_n = -C(\psi_{n-1} - 2\psi_n + \psi_{n+1}) - B|\psi_n|^2\psi_n + Q|\psi_n|^4\psi_n. \tag{2}$$

In the context of continuous system, the corresponding non-autonomous nonlinear Schrödinger equation has been investigated for soliton management, such as dispersion and nonlinearity management in optics, and dispersion management in atomic and condensed-matter physics [11].

The rest of this paper is presented as follows. In Sec. 2, we firstly discuss the variational principle for the autonomous QC-DNLS equation (1) as given in [2], and then provide its justification. In Sec. 3, based on the result in Sec 2, we apply the semi-inverse method to formulate the variational principle for the non-autonomous QC-DNLS equation (2). Finally, we state our conclusion in Sec. 4.

II. Variational Principle For The Autonomous System

In order to construct the variational principle for the non-autonomous QC-DNLS equation (2), let us first consider the variational principle for the corresponding autonomous version, i.e. Eq. (1), which is provided in the following lemma.

Lemma 1. [2] *The variational principle for Eq. (1) is given by :*

$$\mathcal{L} = \sum_{n=-\infty}^{\infty} \frac{i}{2}\left(\psi_n^*\dot{\psi}_n - \psi_n\dot{\psi}_n^*\right) + C(\psi_n^*\psi_{n+1} + \psi_n\psi_{n+1}^* - 2|\psi_n|^2) + \frac{B}{2}|\psi_n|^4 - \frac{Q}{3}|\psi_n|^6, \tag{3}$$

where ψ_n^* represents complex conjugate of ψ_n .

Proof. The variational principle for Eq. (1) is justified by substituting Eq. (3) into the following Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial \psi_n^*} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\psi}_n^*} = 0, \tag{4}$$

from which Eq. (1) is obtained. Note that for any $n = m \in \mathbb{Z}$, the Lagrangian (3) can be written as :

$$\begin{aligned}\mathcal{L} = \cdots &+ \frac{i}{2}\left(\psi_{m-1}^*\dot{\psi}_{m-1} - \psi_{m-1}\dot{\psi}_{m-1}^*\right) + C(\psi_{m-1}^*\psi_m + \psi_{m-1}\psi_m^* - 2|\psi_{m-1}|^2) + \frac{B}{2}|\psi_{m-1}|^4 - \frac{Q}{3}|\psi_{m-1}|^6 \\ &+ \frac{i}{2}\left(\psi_m^*\dot{\psi}_m - \psi_m\dot{\psi}_m^*\right) + C(\psi_m^*\psi_{m+1} + \psi_m\psi_{m+1}^* - 2|\psi_m|^2) + \frac{B}{2}|\psi_m|^4 - \frac{Q}{3}|\psi_m|^6 \\ &+ \cdots. \end{aligned} \tag{5}$$

From Eq. (5), we have

$$\frac{\partial \mathcal{L}}{\partial \psi_m^*} = \frac{i}{2}\dot{\psi}_n + C(\psi_{n+1} - 2\psi_n + \psi_{n+1}) + B|\psi_n|^2\psi_n - Q|\psi_n|^4\psi_n$$

and

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\psi}_m^*} = \frac{i}{2}\dot{\psi}_n.$$

After replacing m by n and upon substituting the above results into Eq. (4), one will produce Eq. (1) as expected.

III. Variational Principle For The Non-Autonomous System

To formulate the variational principle for Eq. (2), we follow the semi-inverse method [6] by introducing $f(t)$, $g(t)$ and $h(t)$, which are unknown functions of t, into Eq. (3) as follows :

$$\mathcal{L} = \sum_{n=-\infty}^{\infty} f(t)\frac{i}{2}\left(\psi_n^*\dot{\psi}_n - \psi_n\dot{\psi}_n^*\right) + g(t)C\left(\psi_n^*\psi_{n+1} + \psi_n\psi_{n+1}^* - 2|\psi_n|^2\right)$$

$$+h(t)\left(\frac{B}{2}|\psi_n|^4 - \frac{Q}{3}|\psi_n|^6\right) + F\left(\psi_n, \psi_n^*, \dot{\psi}_n, \dot{\psi}_n^*, \dots\right), \tag{6}$$

where F is an unknown function of ψ_n, ψ_n^*, and/or its derivatives. Substituting Eq. (6) into the Euler–Lagrange equation (4) yields

$$if(t)\dot{\psi}_n = -g(t)C(\psi_{n-1} - 2\psi_n + \psi_{n+1}) - h(t)(B|\psi_n|^2\psi_n - Q|\psi_n|^4\psi_n) - \frac{i}{2}f'(t)\psi_n - \frac{\delta F}{\delta\psi_n^*}, \tag{7}$$

where $\frac{\delta F}{\delta\psi_n^*}$ is called variational derivative which is defined as

$$\frac{\delta F}{\delta\psi_n^*} = \frac{\partial F}{\partial\psi_n^*} - \frac{d}{dt}\frac{\partial F}{\partial\dot{\psi}_n^*} + \frac{d^2}{dt^2}\frac{\partial F}{\partial\ddot{\psi}_n^*} + \cdots.$$

By comparing Eq. (7) with Eq. (2), one will obtain the following results

$$f(t) = t, \qquad g(t) = 1, \qquad h(t) = 1, \qquad \frac{\delta F}{\delta\psi_n^*} = -\frac{i}{2}f'(t)\psi_n = -\frac{i}{2}\psi_n\,, \tag{8}$$

from which we can identify F in the form

$$F = -\frac{i}{2}|\psi_n|^2. \tag{9}$$

Finally, we obtain the variational principle for Eq. (2) which is given by

$$\mathcal{L} = \sum_{n=-\infty}^{\infty} \frac{it}{2}\left(\psi_n^*\dot{\psi}_n - \psi_n\dot{\psi}_n^*\right) + C\left(\psi_n^*\psi_{n+1} + \psi_n\psi_{n+1}^* - 2|\psi_n|^2\right) + \frac{B}{2}|\psi_n|^4 - \frac{Q}{3}|\psi_n|^6 - \frac{i}{2}|\psi_n|^2.$$

IV. Conclusion

In this paper, we have formulated variational principle for a non-autonomous cubic-quintic discrete nonlinear Schrödinger equation (2) by employing the semi-inverse method. Formulation of the variational principle was performed by introducing integrating factors and an unknown function into the existing variational principle of the autonomous version of the equation. From the calculation process, we confirmed the effectiveness and conciseness of the method.

References

[1] Kevrekidis P. G. 2009. Discrete Nonlinear Schrödinger Equation: Mathematical Analysis,Numerical Computations and Physical Perspectives. Berlin: Springer.

[2] Chong, C., Pelinovsky, D. E., and Schneider, G. 2011. Variational approximation of bifurcation of asymmetric solitons in cubic quintic nonlinear Schrödinger lattice. Discrete and Continuous Dynamical Systems Series. S. 4(5), 1019-1031.

[3] Boudebs, G., Cherukulappurath, S., Leblond, H., Troles, J., Smektala, F and Sanchez, F. 2003. Experimental and theoretical study of higher-order

nonlinearities in chalcogenide glasses. Opt. Commun. 219 , 427-433.

[4] Cassel, K. W. 2003. Variational methods with applications in science and engineering. Cambridge: Cambridge University Press.

[5] Khater, A. H., Moussa, M. H. M and Abdul-Aziz, S. F. 2003. Invariant Variational Principles and Conservation Laws for Some Nonlinear Partial Differential Equations with Constant Coefficients–II. Chaos, Solitons & Fractals. 15, 1–13.

[6] He, Ji-Huan. 1997. Semi-inverse method of establishing generalized variational principles for fluid mechanics with emphasis on turbomachinery aerodynamics. Int. J. Turbo Jet-Engines 14(1), 23–8.

[7] Xu, L. 2006. Variational Principles for Coupled Nonlinear Schrödinger Equations. Physics Letters A. 359, 627-629.

[8] Yao, L., Yang, Yun-Jie and Zhou, Xing-Wei. 2013. A Note on the Semi-Inverse Method and a Variational Principle for the Generalized KdV-mKdV Equation. Abstract and Applied Analysis, Vol. 2013 Article ID 931643.

[9] He, Ji-Huan. 2004. Variational Princple for Some Nonlinear Partial Differential Equations with Variable Coefficients. Chaos, Solitons and Fractals. 19, 847-851.

[10] Laswinda, E., Syafwan, M., Putri, A. R. 2019. Variational Principle for Traveling Waves in a Modified Kuramoto-Sivashinsky Equation. International Journal of Progressive Sciences and Technologies, Vol. 17 No. 1, 31-33.

[11] Malomed, B. A. 2006. Soliton Management in Periodic Systems. New York: Springer.